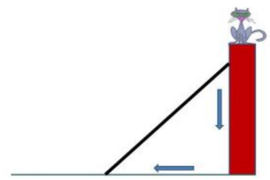
Justo has a twelve-foot ladder leaning against a vertical wall as shown below. Mr. Hipsickles scampers up the ladder, causing the base of the ladder to slide away from the wall.



Problem 1. Let x represent the measures of the quantity "the horizontal distance in feet from the base of the ladder to the wall" and let y represent the measures of the quantity "the vertical distance in feet from the top of the ladder to the ground." In Problem 6 of Investigation1, you showed that in the interval [0,12] we have

$$y = f(x) = \sqrt{144 - x^2}$$

Part (c). Use a graphing calculator to sketch the graph of your function f in the viewing window defined by $0 \le x \le 12$ and $0 \le y \le 12$. What is the meaning of the point $(10, f(10))$?

Part (d). Use the zoom feature on the graphing calculator to zoom in on the graph of the function f, keeping the point $\left(10, f(10)\right)$ at the center of the zoom. If you zoom in several times, what do you notice about the graph of the function f?

Local Linearity

Let $y = f(x)$ be a function. We say the function f is *locally linear near* $x = a$ provided the graph of the function f looks more and more like a nonvertical straight line as we "zoom in" on the point $\left(a, f(a)\right)$.

This straight line, when it exists, is called the *tangent line* to the graph of the function f at the point $\left(a, f(a)\right)$. The point $\left(a, f(a)\right)$ under consideration is called the *point of tangency*.

Problem 2. Consider the function $y = f(x) = 1 + |x - 1|$.

Part (a). Enter this function into your graphing calculator and use your calculator to sketch its graph in the viewing window $-1 \le x \le 2$, $0 \le y \le 3$. Do you think that the function f is locally linear at the point $(0, 2)$? Explain.

Part (b). Is the function f locally linear at the point $(1, 1)$? Explain.

Problem 3. Let's consider the function $y = f(x) = \cos(x)$ in the interval $[0,5]$.

Part (a). What is the value of $f(2.5)$?

Part (b). What is the value of $f'(2.5)$?

Part (c). On your graphing calculator, sketch the graph of the function f with the viewing window set to $0 \le X \le 5$ XScl $= 0.5$ and $-2 \le Y \le 1$ YScl $= 0.5$. Also graph the linear function

$$y = T(x) = f'(2.5)[x - 2.5] + f(2.5)$$

What are some things you notice?

Part (d). Use the zoom feature on your graphing calculator to zoom in on the point $\big(2.5, f(2.5)\big)$ several times. What do you notice?

Local Linearity and Differentiability

Suppose that $y = f(x)$ is a function. The following two statements are always true.

- If f is locally linear near an input value $x = a$, then f is differentiable at $x = a$.
- If f is differentiable at an input value $x = a$, then f is locally linear near $x = a$.

Furthermore, if f is locally linear near an input value $x = a$, then the tangent line to the graph of f at the point $\big(a, f(a)\big)$ is given by the function

$$y = T(x) = f'(a)[x - a] + f(a)$$

If a function $y = f(x)$ is differentiable at an input value $x = a$, then we know

1. For any value of h, we have $f'(a) \approx \text{ARC}_h(a)$.
2. This approximation improves as the values of h approach 0.

If we know the value of the outputs $f'(a)$ and $f(a)$, then we can use this information to approximate the value of $f(a + h)$; and we would expect this approximation to be very accurate if $|h|$ is small. In particular,

$$f'(a) \approx \frac{f(a+h) - f(a)}{h} \quad \Rightarrow \quad f(a+h) \approx f(a) + f'(a) \cdot h$$

This observation is the basis for *Euler's Method* --- a technique for approximating the graph of a function if we happen to know its derivative.

Problem 4. Oil is flowing from a ruptured pipeline deep below the Gulf of Mexico. Let V represent measures of the quantity "the volume of oil that has leaked from the pipeline, measured in gallons," and let t represent measures of the quantity "the time passed since the leak was detected, measured in hours." Assume that $V = f(t)$ in the interval $[0, +\infty)$. Scientists estimate that 10 gallons of oil had escaped before the leak was detected, and remote monitors sampling the flow have determined that

$$\frac{df}{dt}(t) = 50 \cdot \frac{t + \cos(\pi t)}{2 - \sin(\pi t)}$$

Part (a). We know that $f(0) = 10$ gallons. What is the value of $f'(0)$?

Part (b). For any nonzero value of h, we know

$$f(0 + h) \approx f(0) + f'(0) \cdot h$$

Use this formula to estimate the value of $f(0.25)$.

Part (c). Do you think this estimate for $f(0.25)$ is accurate? Explain your thinking.

Part (d). Let's assume that our estimate for $f(0.25)$ is *exact*. For any nonzero value of h, we know

$$f(0.25 + h) \approx f(0.25) + f'(0.25) \cdot h$$

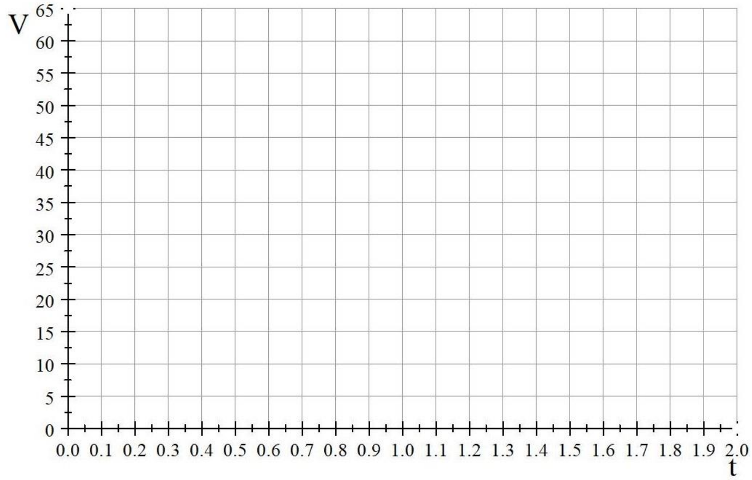
Use this formula to estimate the value of $f(0.50)$.

Part (e). Proceeding in this manner, fill in the following table.

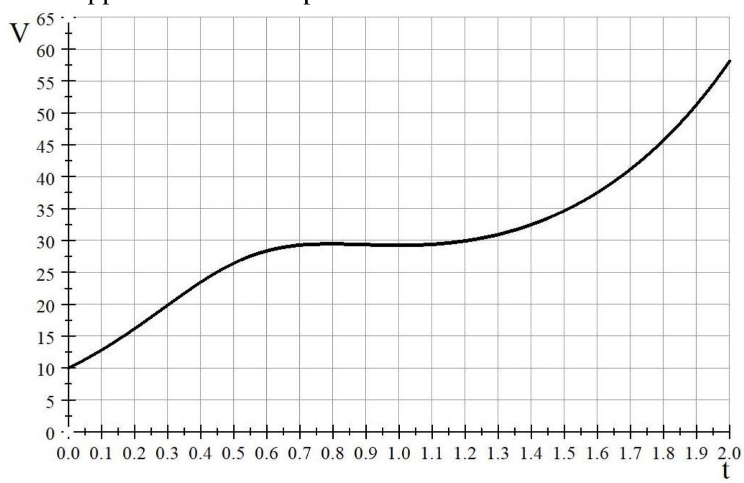
Value of t	0.25	0.5	0.75	1.0	1.25	1.5	1.75	2
Approxmiate Value of $f(t)$								

We do not know how the function f coordinates the values of t and V in the time intervals between sample times. (We could use smaller time intervals, but the problem persists in those intervals as well.) Therefore, we simply use straight line segments to "fill in" the covariation between our data points, thereby creating a *piecewise linear* graph. (Such graphs are also known as *polygonal curves*.)

Problem 5. Use the data points in your table to construct a piecewise-linear graph that approximates the graph of the function $V = f(t)$ in the interval $0 \leq t \leq 2$.

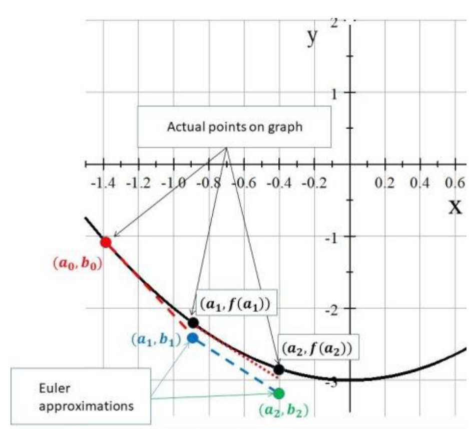


The trace below shows the *actual* graph of the function $V = f(t)$ in the interval $0 \leq t \leq 2$. How well does your piecewise-linear approximation compare?



Euler's Method

Suppose we know a derivative function $r = f'(x)$ and one point (a_0, b_0) that lies on the graph of the function $y = f(x)$. We can approximate the graph of the function $y = f(x)$ using the following procedure.

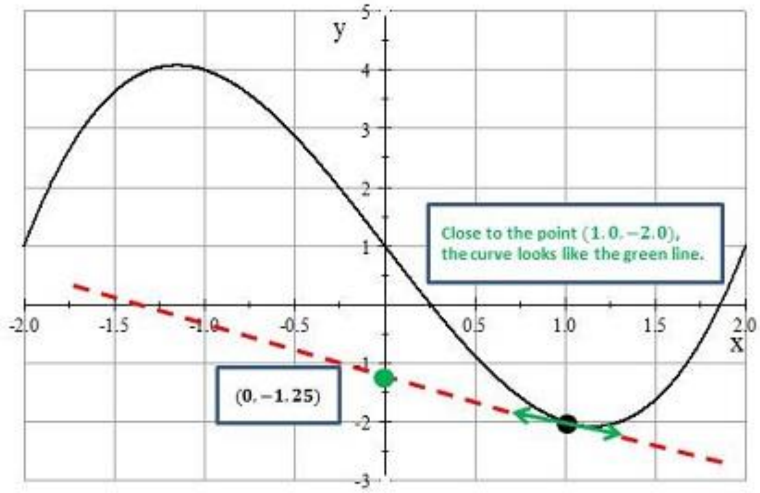


1. Let h be a fixed, relatively small number. (The smaller the better.)
2. Let $a_1 = a_0 + h$. (We are considering an input value a little to the right or left of $x = a_0$.)
3. We know (a_0, b_0) is a point on the graph. (In symbols, we know $b_0 = f(a_0)$).
4. Let $b_1 = b_0 + f'(a_0) \cdot h$. The point (a_1, b_1) lies on the *tangent line* to the graph of the function f at the point (a_0, b_0); therefore, if h is small, we know $y = b_1$ is a good approximation to the value $y = f(a_1)$. The point (a_1, b_1) is therefore *approximately* on the graph of f.
5. Let $a_2 = a_1 + h$ and consider the value $b_2 = b_1 + f'(a_1) \cdot h$.
6. The point (a_2, b_2) lies on a line that is *parallel* to the tangent line to the graph of f at the point $(a_1, f(a_1))$. Therefore, the we know $f(a_2) \approx b_2$. The point (a_2, b_2) is therefore *approximately* on the graph of f. (This approximation won't be as good as the first approximation, however.)
7. Repeat steps 5 and 6 as often as you wish to create a sequence of points which lie approximately on the graph of f, remembering that the approximations probably get progressively worse the more your repeat.

Homework.

Problem 1. Consider a function $y = f(x)$. Suppose the tangent line to the graph of the function f at the point $(-2, f(-2))$ is given by the formula $y = T(x) = -4(x + 2) - 8$. Based on this information, what is the instantaneous rate of change with respect to x for the function f at the input value $x = -2$?

Problem 2. Consider the graph of a function $y = f(x)$ shown below. This graph is locally linear near the input value $x = 1$, because if we focus on the graph *very* close to the point $(1.0, -2.0)$, then the graph is *almost* a straight line.



Part (a). Construct the point-slope formula for the approximate tangent line to the graph at the point $(1.0, -2.0)$.

Part (b). Carefully draw an approximate tangent line to the graph of the function f at the point $(-1.5, 3.75)$, then use your sketch to construct the point-slope formula for the line.

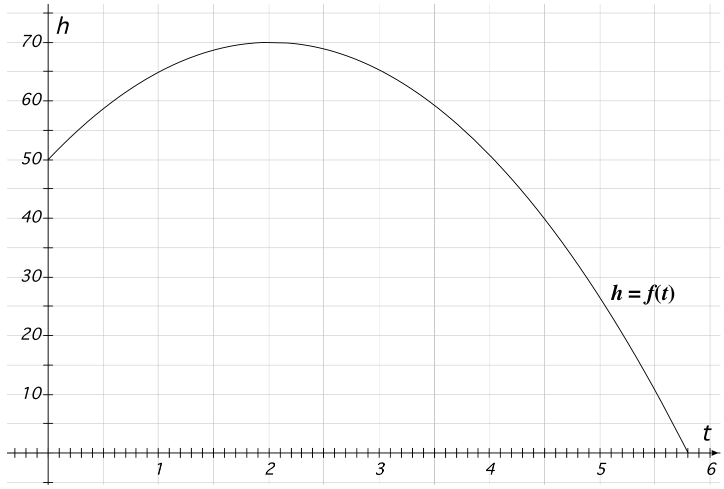
Problem 3. Consider a function $m = f(n)$ and suppose that $f(3.8) = 2.75$. Suppose we also know

$$\lim_{h \to 0} \frac{f(3.8 + h) - f(3.8)}{h} = 1.83$$

What is the formula for the tangent line to the graph of f at the point $(3.8, f(3.8))$?

Problems 3 - 5 refer to the following scenario.

Toby is standing on top of a five-story building. He leans over the edge of the building and tosses a penny vertically into the air, then steps back and watches it rise and then fall to the ground. The graph on the below shows the relationship between the values of the height h in feet of the penny above the ground and the values of time t in seconds since Toby tossed the penny.



Problem 4. Carefully draw the tangent line to the graph of the function f at the point $(3,65.30)$.

Part (a). Construct the point-slope formula for the straight line you drew.

Part (b). What is the approximate value of $f'(3)$?

Problem 5. Based on the graph, at what input value x would we expect to have $f'(x) = 0$?

Problem 6. Based on the graph, on what input interval would we expect to have $f'(x) < 0$?

Problem 7. Consider the function $z = f(q) = 4q^2 - 3q + 5$. In Homework Problem 3 of Investigation 8, you showed that $f'(q) = 8q - 3$.

Part (a). For what values of q would the tangent line to the graph of f be horizontal?

Part (b). For what values of q would the tangent line to the graph of f have a constant rate of change $m = -4$?

Problem 8. Consider the function $y = f(x) = \sqrt{x}$. In Investigation 7, you showed that

$$f'(x) = \frac{1}{2\sqrt{x}}$$

Part (a). Is there any input value x where the tangent line to the graph of f is horizontal?

Part (b). Are there any input values x where the tangent line to the graph of f has a constant rate of change $m = 3$?

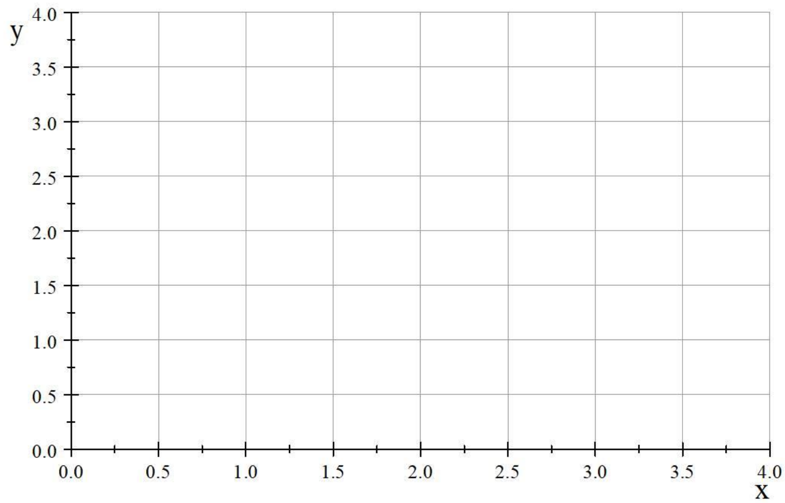
Problem 9. Suppose we know that the derivative function for $y = f(x)$ is given by the formula

$$f'(x) = \frac{2-x}{1+x}$$

Part (a). Suppose we also know that $f(0) = 1$. Let $h = 0.5$ and use Euler's Method to fill in the following table.

Value of x	$a_1 = 0.5$	$a_2 = 1.0$	$a_3 = 1.5$	$a_4 = 2.0$	$a_5 = 2.5$	$a_6 = 3.0$	$a_7 = 3.5$	$a_8 = 4.0$
Approxmiate Value of $f(x)$								

Part (b). Construct a piecewise linear curve that approximates the graph of the function f in the input interval $0 \le x \le 4$.



Answers to the Homework.

Problem 1. The instantaneous rate of change is $f'(-2) = -4$.

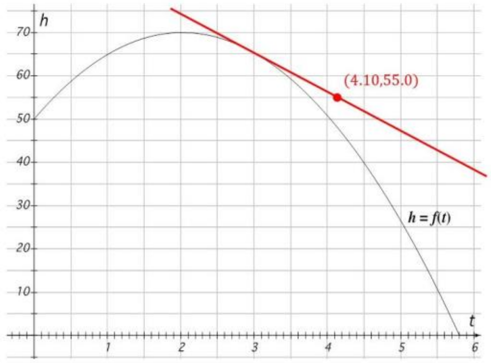
Problem 2.
Part (a). The tangent line to the graph of the function at the point $(1, -2)$ is $T(x) \approx -0.75(x - 1) - 2$.

Part (b). Measures of the slope for this line will vary slightly; however, the tangent line to the graph of the point $(-1.5, 3.75)$ will pass very *close* to the point $(-1.0, 5.0)$. Using these points, we find that the tangent line to the graph of the function at the point $(-1.5, 3.75)$ is

$$T(x) \approx 2.5(x + 1.5) + 3.75.$$

Problem 3. The point-slope formula is $y = T(x) = 1.83(x - 3.8) + 2.75$.

Problem 4. Here is an approximation to the desired tangent line.



The point of tangency is given to be (3.0,65.3); therefore, the constant rate of change with respect to x for this line is

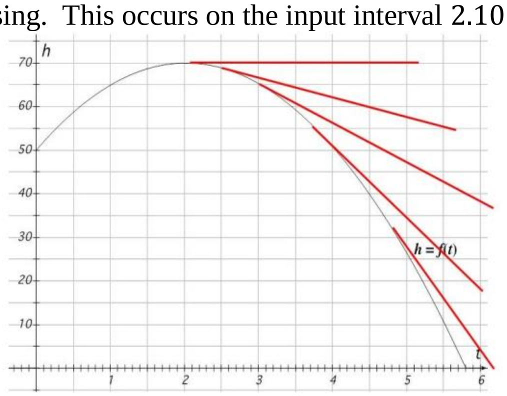
$$m = \frac{55.0 - 65.3}{4.10 - 3.00} \approx -9.364$$

This tells us that $f'(3) \approx -9.364$, and the point-slope formula for the tangent line is

$$y = T(x) \approx -9.364(x - 3) + 65.3$$

Problem 5. The function f has a local maximum output at the input value $x \approx 2.10$. Our work with average rate of change functions tells us that the output of $r = \text{ARC}_h(x)$ will change from positive to negative *near* this input value. The input value where this change occurs will approach $x = 2.10$ as the values of h approach 0. Therefore, it stands to reason that we will have that $f'(x) = 0$ at the input value $x \approx 2.10$.

Problem 6. Looking at the graph, it appears that the tangent lines to the graph of f will have negative slope when the graph is decreasing. This occurs on the input interval $2.10 < x$.



Problem 7.
Part (a). Observe

$$f'(q) = 0 \quad \Longrightarrow \quad 8q - 3 = 0 \quad \Longrightarrow \quad q = \frac{3}{8}$$

There is only one input value where the tangent line to the graph of f is horizontal.

Part (b). Observe

$$f'(q) = -4 \quad \Longrightarrow \quad 8q - 3 = -4 \quad \Longrightarrow \quad q = -\frac{1}{8}$$

There is only one input value where the tangent line to the graph of f has slope $m = -4$.

Problem 8.
Part (a). No, there are no such input values. The equation $f'(x) = 0$ has no solutions.

Part (b). Observe

$$f'(x) = 3 \quad \Longrightarrow \quad \frac{1}{2\sqrt{x}} = 3 \quad \Longrightarrow \quad \frac{1}{6} = \sqrt{x} \quad \Longrightarrow \quad \frac{1}{36} = x$$

Problem 9. Suppose we know that the derivative function for $y = f(x)$ is given by the formula

$$f'(x) = \frac{2 - x}{1 + x}$$

Part (a). Suppose we also know that $f(0) = 1$. Let $h = 0.5$ and use Euler's Method to fill in the following table.

Remember, $f(a_{n+1}) \approx f(a_n) + h \cdot f'(a_n)$. Therefore, once you know $f(a_n)$, you can use this information to approximate $f(a_{n+1})$.

1. We know $f(a_0) = 1$. Therefore, $f(a_1) \approx f(a_0) + 0.5 \cdot f'(a_0) = 2.00$.
2. We know $f(a_1) \approx 2.00$. Therefore, $f(a_2) \approx f(a_1) + 0.5 \cdot f'(a_1) = 2.50$.
3. We know $f(a_2) \approx 2.50$. Therefore, $f(a_3) \approx f(a_2) + 0.5 \cdot f'(a_2) = 2.75$.
4. We know $f(a_3) \approx 2.75$. Therefore, $f(a_4) \approx f(a_3) + 0.5 \cdot f'(a_3) \approx 2.85$.
5. We know $f(a_4) \approx 2.85$. Therefore, $f(a_5) \approx f(a_4) + 0.5 \cdot f'(a_4) \approx 2.85$.
6. We know $f(a_5) \approx 2.85$. Therefore, $f(a_6) \approx f(a_5) + 0.5 \cdot f'(a_5) \approx 2.78$.

Value of x	$a_1 = 0.5$	$a_2 = 1.0$	$a_3 = 1.5$	$a_4 = 2.0$	$a_5 = 2.5$	$a_6 = 3.0$	$a_7 = 3.5$	$a_8 = 4.0$
Approxmiate Value of $f(x)$	2.00	2.50	2.75	2.85	2.85	2.78	2.655	2.49