The various processes we use to determine the derivative function for f are known collectively as *differentiation*.

Differentiating a function $y = f(x)$ amounts to determining the derivative function $r = f'(x)$. To do this, we must find a way to identify the function produced by the limiting process

$$\frac{df}{dx}(x) = \lim_{h\to 0} ARC_h(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

When we want to differentiate a function of the input variable x, it is common to use the following special symbol to represent the process involved:

$$\frac{d}{dx}[\quad]$$

This notation is read "construct the derivative function with respect to the variable x."

In previous investigations, you developed the formula for the derivative function of a number of specific functions. For example, in Investigations 6 and 7, you developed the following specific derivative formulas.

Specific Derivative Formulas From Investigations 6 and 7

$$\frac{d}{dx}[\sin(x)] = \cos(x) \qquad \frac{d}{dx}[\cos(x)] = -\sin(x) \qquad \frac{d}{dx}[e^x] = e^x \qquad \frac{d}{dx}[\ln(x)] = \frac{1}{x}$$

(We established the formulas above by looking at the graphs of average rate of change functions.)

$$\frac{d}{dx}[c] = 0 \qquad \frac{d}{dx}[x] = 1 \qquad \frac{d}{dx}[x^2] = 2x \qquad \frac{d}{dx}[x^3] = 3x^2 \qquad \frac{d}{dx}[\sqrt{x}] = \frac{1}{2\sqrt{x}} \qquad \frac{d}{dx}\left[\frac{1}{x}\right] = -\frac{1}{x^2}$$

(We established the formulas above by using algebraic manipulation on the average rate of change formulas.)

We will refer to these as *specific derivative formulas*, because they are formulas for the derivative functions of specific functions. We will develop a number of specific derivative formulas in the next few investigations.

There is another type of derivative formula that does not refer to specific functions. Formulas of this type --- known as *general derivative rules* --- tell us how to construct the derivative function for functions that are built up from other functions whose derivative formulas we already know.

Problem 1. Suppose that k is a fixed real number, and suppose $y = f(x)$ is differentiable in an interval I. Let $y = g(x)$ be defined by the formula $g(x) = k \cdot f(x)$.

Part (a). Use algebraic manipulation to show $\text{ARC}_h(x) = k \cdot \dfrac{f(x+h)-f(x)}{h}$ for the function g.

Part (b). What does Part (a) tell you about the function the process $\lim_{h\to 0} \text{ARC}_h(x)$ produces?

Problem 2. Suppose that f and g are both differentiable functions in an interval I. Let $y = A(x)$ be a new function defined by the formula $y = A(x) = f(x) + g(x)$.

Part (a). Use algebraic manipulation to show that the average rate of change functions for the function A are defined by the formula

$$r = \text{ARC}_h(x) = \left(\frac{f(x+h) - f(x)}{h}\right) + \left(\frac{g(x+h) - g(x)}{h}\right)$$

Part (b). What does Part (a) tell you about the function the process $\lim_{h\to 0} \text{ARC}_h(x)$ produces?

Sum and Constant Multiple Rules

Suppose $y = f(x)$ and $y = g(x)$ are differentiable functions in an interval I, and suppose that k is any constant.

Constant Multiple Rule

- The constant multiple function $y = k \cdot f(x)$ is also differentiable in this interval, and

$$\frac{d}{dx}[k \cdot f(x)] = k \cdot f'(x) = k \cdot \frac{df}{dx}(x)$$

Sum Rule

- The function sum $y = f(x) + g(x)$ is also differentiable in this interval, and

$$\frac{d}{dx}[f(x) + g(x)] = f'(x) + g'(x) = \frac{df}{dx}(x) + \frac{dg}{dx}(x)$$

Example 1. What is the formula for the derivative function for $y = f(u) = 3\sqrt{u} + 8$?

Solution. Observe that

$$\frac{d}{du}[3\sqrt{u} + 8] = \frac{d}{du}[3\sqrt{u}] + \frac{d}{du}[8]$$ Apply Sum Rule for Derivatives

$$= 3\frac{d}{du}[\sqrt{u}] + \frac{d}{du}[8]$$ Apply Constant Multiple Rule for Derivatives

$$= \frac{3}{2\sqrt{u}} + 0$$ Apply Specific Derivative Formulas

Problem 2. Differentiate the function $b = f(q) = \sqrt{3} \cdot \cos(q) - 5q + e$. Indicate how the sum and constant multiple rules are used in the process.

Problem 3. For any fixed positive real number a, the Base-a logarithm in the interval $0 < x < \infty$ is defined by the formula

$$y = \log_a(x) = \frac{\ln(x)}{\ln(a)}$$

Compute $\frac{d}{dx}[\log_a(x)]$.

Problem 4. Lucinda launches a drone off the top of a building. Let t represent measures of the quantity "the time passed since Lucinda launched the drone, measured in minutes," and let a represent measures of the quantity "the distance between the ball and the ground, measured in feet." Suppose that

$$a = f(t) = t^3 - 9t + 20$$

in the interval $0 \le t \le 5$ seconds.

Part (a). What are the input values in this interval where the instantaneous rate of change in the output of f with respect to t is 0?

Part (b). What is the significance of these input values in the context of this problem?

Problem 4. Let $y = f(b) = 3b - \sqrt{b}$ in the interval $-\infty < b < \infty$. Determine the input values where the instantaneous rate of change in the output of f with respect to b is equal to 1.

Problem 5. Suppose that the derivative function for a function $y = f(x)$ is given by the function $r = f'(x) = x^2 - x - 2$ in the interval $-3 \le x \le 3$.

Part (a). There are two x-intercepts for the derivative function (input values x such that $f'(x) = 0$) in this input interval. What are these values?

Part (b). Choose any input value $x = a$ you like strictly in between the two x-intercepts. Is $f'(a)$ positive or negative?

Part (c). If you choose a different input value $x = b$ strictly in between the two x-intercepts, do you think $f'(b)$ would have a different sign from $f'(a)$? Explain your thinking.

Homework.

Use the sum and constant multiple rules, along with the specific derivative formulas to differentiate the following functions.

(1) $f(x) = e^5$

(2) $g(z) = 5.2z + 2.3$

(3) $h(t) = 2t^3 - 4t + \sqrt{3}$

(4) $f(x) = \frac{2x^{-1}}{5} - 3\ln(x)$

(5) $g(z) = z - \frac{1}{z} + 2\sin(z)$

(6) $h(t) = \sqrt{t} - \frac{2}{t} + \pi\ln(t)$

(7) $f(x) = 4x + 2\sqrt{5x}$

(8) $g(z) = \sqrt{3} \cdot e^z + \frac{\pi}{z} - 4\log_7(t)$

(9) $h(t) = \log_3(t) - 8t^3 - 8\cos(t)$

Problem 10. Consider the function $y = f(x) = 2x^3 + x$. What is the value of $f'(1)$?

Problem 11. Consider the function $y = f(x) = 4x^3 - 36x + 1$ in the interval $-2 \le x \le 1$. Determine all of the input values in this interval where the instantaneous rate of change in the output of f with respect to x is 0.

Problem 12. Consider the function $y = f(x) = \frac{2}{x} - 3x + 8$ in the interval $-2 \leq x \leq 2$. Determine all of the input values in this interval where the instantaneous rate of change in the output of f with respect to x is -4.

Answers to the Homework.

(1) $f'(x) = 0$

(2) $g'(z) = 5.2$

(3) $h'(t) = 6t^2 - 4$

(4) $f'(x) = -\left(\frac{2}{5}\right)x^{-2} - \frac{3}{x}$

(5) $g'(z) = 1 + z^{-2} + 2\cos(z)$

(6) $h'(t) = \frac{1}{2\sqrt{t}} + \frac{2}{t^2} + \frac{\pi}{t}$

(7) $f'(x) = 4 + \sqrt{\frac{5}{x}}$

(8) $g'(z) = 3e^x - \pi z^{-2} - \frac{4}{x\ln(7)}$

(9) $h'(t) = \frac{1}{x\ln(3)} - 24t^2 + 8\sin(t)$

Problem 10. First, note that $f(1) = 3$ and $f'(x) = 6x^2 + 1$. Consequently, $f'(1) = 7$.

Problem 11. First, note that $f'(x) = 12x^2 - 18$. Now,

$$f'(x) = 0 \quad \Longrightarrow \quad 2x^2 - 3 = 0 \quad \Longrightarrow x = \pm\sqrt{\frac{3}{2}}$$

Only $x = -\sqrt{3/2}$ lies in the specified input interval.

Problem 12. First, note that we want $f'(x) = -4$. Now, observe that

$$f'(x) = -\left(3 + \frac{2}{x^2}\right)$$

$$f'(x) = -4 \quad \Longrightarrow \quad \left(3 + \frac{2}{x^2}\right) = 4 \quad \Longrightarrow 2 = x^2$$

This last equation has two solutions, namely $x = \pm\sqrt{2}$. Both of these solutions lie in the specified interval.