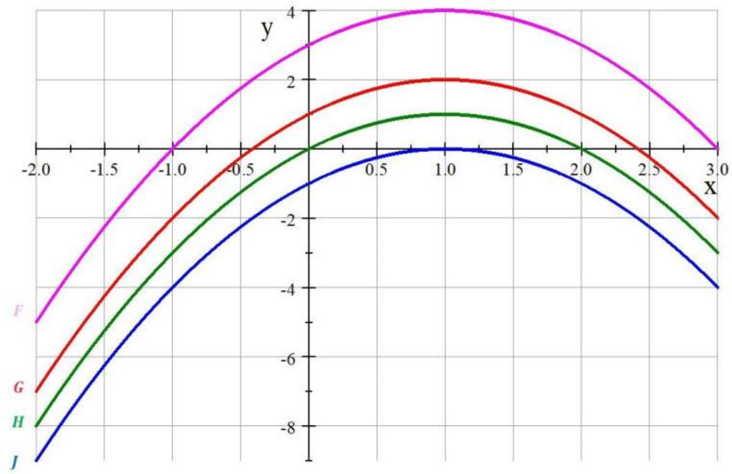
In the previous investigation, we used Euler's Method to approximate the graph of a function whose *derivative* is a specified function. Whenever we have found a function $y = F(x)$ whose derivative function is a specified function $r = f(x)$, we have identified an *antiderivative* for the function f. We can always use Euler's method to approximate the graph of an antiderivative, but there are sometimes other ways of identifying them.

> ***First Rule of Antiderivatives***
>
> Let $y = F(x)$ be a function in an interval. Whenever we compute the derivative $r = F'(x)$, we automatically know that F is one antiderivative for F'.

For example, we already know that $y = F(x) = \sin(x)$ is one antiderivative for $r = f(x) = \cos(x)$.

Problem 1. Show that $F(x) = x^3$, $G(x) = x^3 - 1$, and $H(x) = x^3 + 3\sqrt{2}$ are *all* antiderivatives for the function $f(x) = 3x^2$ in the interval $-\infty < x < \infty$.

Problem 2. Consider the functions whose graphs are shown below. Explain why *all* of these functions are antiderivatives for the same function in the interval $-2 \leq x \leq 3$. (Think the tangent line to each of these graphs at the same input value.)



Problem 3. What is one antiderivative function for $f(x) = \cos(x)$? Explain your thinking.

Sometimes, we can apply the general derivative rules *in reverse* to determine antiderivative families. For example, suppose we want to know the antiderivative family for the function $r = f(x) = x^2$.

$\frac{d}{dx}[x^3] = 3x^2 \quad \Rightarrow \quad \left(\frac{1}{3}\right)\frac{d}{dx}[x^3] = x^2$

$\Rightarrow \quad \frac{d}{dx}\left[\frac{1}{3}x^3\right] = x^2$ ***(Apply the Constant Multiple Rule in reverse.)***

This tells us that $F(x) = \frac{1}{3}x^3$ is one antiderivative for $f(x) = x^2$.

Problem 4. Identify three other antiderivatives for $f(x) = x^2$. Explain your strategy.

Problem 5. Use the fact that $\frac{d}{dx}\left[\sqrt{x}\right] = \frac{1}{2\sqrt{x}}$ to find one antiderivative for the function $f(x) = x^{-1/2}$.

Problem 6. Consider the function $y = g(x) = Kx$, where K is any fixed real number.

Part (a). Construct the formula for $r = g'(x)$.

Part (b). Use your answer to Part (a) to determine two antiderivatives for the constant output function $f(x) = K$.

Two General Antiderivative Rules

Suppose that $r = f(x)$ and $r = g(x)$ are functions in an interval, and suppose that K is a fixed number.

- **Anti-Constant-Multiple Rule:**
  If $y = F(x)$ is any antiderivative for f, then $y = K \cdot F(x)$ is one antiderivative for $r = K \cdot f(x)$.
- **Anti-Sum Rule:**
  If $y = F(x)$ is any antiderivative for f and $y = G(x)$ is any antiderivative for g, then $y = F(x) + G(x)$ is one antiderivative for $r = f(x) + g(x)$.

Problem 7. Use the general antiderivative rules to identify one antiderivative for the function

$$r = f(x) = 4\cos(x) - \frac{1}{\sqrt{x}} + \sqrt[3]{5}$$

Problem 8. Use the general antiderivative rules to identify one antiderivative for the function

$$r = g(t) = \frac{3}{x^2} - \pi\sin(x) + x^2 - e$$

As mentioned at the beginning of this investigation, we can always use Euler's Method to create piecewise linear graphs that approximate one antiderivative for a function. The accuracy of the approximation will, of course, depend on how small we set the increment in our input variable.

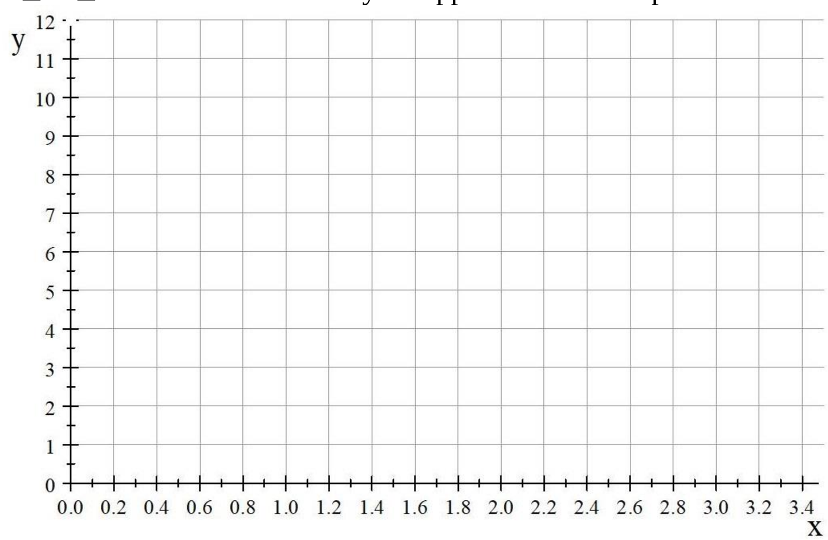
Problem 9. We know that $y = F(x) = x^2 + 3\cos(x)$ is one antiderivative for the function

$$r = f(x) = 2x - 3\sin(x)$$

Part (a). Starting at the point $(0,3)$, use Euler' Method to fill in the following table.

Value of x	0.0	0.5	1.0	1.5	2.0	2.5	3.0	3.5
Approximate Value of $F(x)$	3.0							

Part (b). Use the data in your table to construct a piecewise linear graph that approximates the graph of F in the interval $0 \le x \le 3.5$. How accurate is your approximation compared to the actual graph?



Homework.

Problem 1. By differentiating the function F, show that $y = F(x) = x^2 - 4\sqrt{x} + 2e$ is one antiderivative for the function

$$r = f(x) = 2x - \frac{2}{\sqrt{x}}$$

Problem 2. Let $y = F(x) = 2e^x + \ln(x)$ in the interval $0 < x < \infty$. By differentiating F, show that it is one antiderivative for the function $r = f(x) = 2e^x + x^{-1}$ in this interval.

Use the general antiderivative rules to identify one antiderivative for each of the following functions.

(3) $f(x) = 2e^x - 3\cos(x)$ **(4)** $g(t) = 3t + 5t^{-2}$ **(5)** $h(v) = 2 - \frac{1}{\sqrt{v}}$

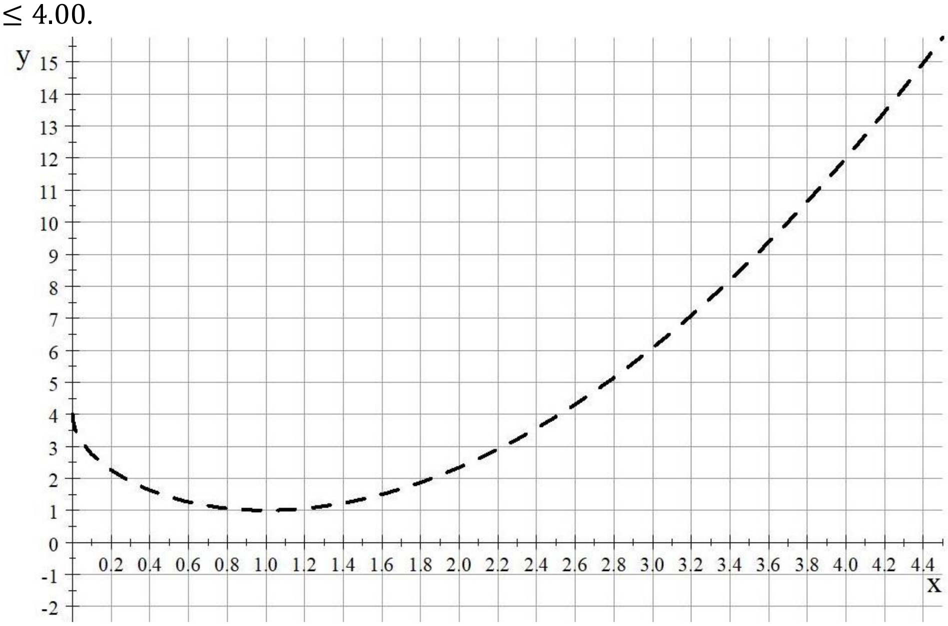
(6) $f(x) = \frac{\pi}{2} - \frac{1}{x}$ $(x > 0)$ **(7)** $g(t) = \frac{\sqrt{2}}{t^2} - 5t^2$ **(8)** $h(v) = e\sin(v) - 4v^2$

Problem 9. Let's consider the function $r = f(x)$ from Problem 1. We know that the function $y = G(x) = 4 + x^2 - 4\sqrt{x}$ is one antiderivative function for the function f. We also know that $(4,12)$ is a point on the graph of G.

Part (a). Use Euler's Method to fill in the following table. (Note that $h = -0.40$ in this case.)

Value of x	4.00	3.60	3.20	2.80	2.40	2.00	1.60	1.20	0.80	0.40	0.00
Approximate Value of $F(x)$	12.00										

Part (b). The diagram below shows the trace of the actual antiderivative G. On the same grid, use the data in your table to construct a piecewise linear graph that approximates the graph of G in the interval $0.00 \le x \le 4.00$.



Answers to the Homework.

Problem 1. Observe

$$\frac{d}{dx}\left[x^2 - 4\sqrt{x} + 2e\ \right] = \frac{d}{dx}[x^2] - 4\frac{d}{dx}\left[\sqrt{x}\right] + \frac{d}{dx}[2e] = 2x - \frac{4}{2\sqrt{x}} + 0 = 2x - \frac{2}{\sqrt{x}}$$

Problem 2. Observe

$$\frac{d}{dx}[2e^x + \ln(x)\ ] = 2\frac{d}{dx}[e^x] + \frac{d}{dx}[\ln(x)] = 2e^x + \frac{1}{x}$$

Problem 3. We know

$$\frac{d}{dx}[e^x] = e^x \qquad \frac{d}{dx}[\sin(x)] = \cos(x)$$

Therefore, we know that one antiderivative for $f(x) = 2e^x - 3\cos(x)$ is the function

$$F(x) = 2e^x - 3\sin(x)$$

Problem 4. We know

$$\frac{d}{dt}\left[\frac{1}{2}t^2\right] = t \qquad \frac{d}{dt}\left[-\frac{1}{t}\right] = \frac{1}{t^2}$$

Therefore, we know that one antiderivative for $g(t) = 3t + 5t^{-2}$ is the function

$$G(t) = \frac{3}{2}t^2 - \frac{5}{t}$$

Problem 5. We know

$$\frac{d}{dv}[v] = 1 \qquad \frac{d}{dv}\left[2\sqrt{v}\right] = \frac{1}{\sqrt{v}}$$

Therefore, we know that one antiderivative for $h(v) = 2 - \frac{1}{\sqrt{v}}$ is the function

$$H(v) = 2v - 2\sqrt{v}$$

Problem 6. We know

$$\frac{d}{dx}[x] = 1 \qquad \frac{d}{dx}[\ln(x)] = \frac{1}{x}$$

Therefore, as long as $x > 0$, we know that one antiderivative for $f(x) = f(x) = \frac{\pi}{2} - \frac{1}{x}$ is the function

$$F(x) = \frac{\pi}{2}x - \ln(x)$$

The restriction that $x > 0$ is important here because the function $u(x) = x^{-1}$ is defined for negative input, while the function $v(x) = \ln(x)$ is not. Consequently, the function v cannot serve as an antiderivative for the function u when $x < 0$.

Problem 7. We know

$$\frac{d}{dt}\left[-\frac{1}{t}\right] = \frac{1}{t^2} \qquad \frac{d}{dt}\left[\frac{1}{3}t^3\right] = t^2$$

Therefore, we know that one antiderivative for $g(t) = \frac{\sqrt{2}}{t^2} - 5t^2$ is the function

$$G(t) = -\frac{\sqrt{2}}{t} - \frac{5}{3}t^3$$

Problem 8. We know

$$\frac{d}{dv}[-\cos(v)] = \sin(v) \qquad \frac{d}{dv}\left[\frac{1}{3}v^3\right] = v^2$$

Therefore, we know that one antiderivative for $h(v) = e\sin(v) - 4v^2$ is the function

$$H(v) = -e\cos(v) - \frac{4}{3}v^3$$

Problem 9. Note that

$$F(3.60) \approx 12.00 + f(4.00) \cdot (-0.40) = 12 - 0.40\left(2(4.00) - \frac{2}{\sqrt{4.00}}\right) = 9.20$$

$$F(3.20) \approx 12.00 + f(3.60) \cdot (-0.40) = 9.2 - 0.40\left(2(3.60) - \frac{2}{\sqrt{3.60}}\right) \approx 7.09$$

Continuing in this manner, we have

Value of x	4.00	3.60	3.20	2.80	2.40	2.00	1.60	1.20	0.80	0.40	0.00
Approximate Value of $F(x)$	12.00	9.20	7.09	4.98	3.22	2.26	1.23	0.58	0.35	0.60	1.55

Part (b).