In this investigation, we will explore patterns that emerge when we differentiate the *product* of two differentiable functions.

Problem 1. Consider the functions defined by the formulas below in the interval $-\infty < x < \infty$.

$$y = f(x) = 2x^3 - 5x^2 + 4x - 1 \qquad m(x) = x^2 - 2x + 1 \qquad n(x) = 2x - 1$$

Part (a). Construct the formulas for $r = f'(x)$, $r = m'(x)$, and $r = n'(x)$.

Part (b). Show that $f(x) = m(x) \cdot n(x)$.

Part (c). Is it true that $f'(x) = m'(x) \cdot n'(x)$?

Problem 2. Consider the function $y = f(x) = x \cdot \cos(x)$ in the interval $-\infty < x < \infty$.

Part (a). Enter this function as Y_1 in your graphing calculator; turn this function off.

Part (b). Set $L_1 \to \{1.0, 0.5, 0.25, 0.1, 0.01\}$. Set your viewing window to $-\pi \leq X \leq \pi$ XScl $= \pi/2$ and $-4 \leq Y \leq 4$ YScl $= 1$. Graph the five average rate of change functions for Y_1 in this window.

Part (c). Now, add the graph of the function $Y_3 = \cos(x) - x\sin(x)$.

Part (d). Make a conjecture about how $r = f'(x)$ is related to the derivatives of $y = m(x) = x$ and $y = n(x) = \cos(x)$.

Problem 3. Using the functions f, m, and n from Problem 1, test your conjecture.

It would seem that the derivative formula for a product of two functions is *not* the product of the derivatives of the factors. Let's get some insight as to why by looking closely at the average rate of change functions associated with the product and its factors. Remember that the derivative of any function is based on the average rate of change functions associated with that function. Note that

$$\mathrm{ARC}_h^{(f)}(x) = \frac{f(x+h) - f(x)}{h} \qquad \mathrm{ARC}_h^{(m)}(x) = \frac{m(x+h) - m(x)}{h} \qquad \mathrm{ARC}_h^{(n)}(x) = \frac{n(x+h) - n(x)}{h}$$

Since we know that $f(x) = m(x) \cdot n(x)$, we can say more. In particular, we know

$$\mathrm{ARC}_h^{(f)}(x) = \frac{f(x+h) - f(x)}{h} = \frac{m(x+h)n(x+h) - m(x)n(x)}{h}$$

We would expect that the formula for $\text{ARC}_h^{(f)}$ *somehow* involves the formulas for $\text{ARC}_h^{(m)}$ and $\text{ARC}_h^{(n)}$, but exactly how is not obvious from the formula we have just constructed. However, we can *insinuate* $\text{ARC}_h^{(m)}$ and $\text{ARC}_h^{(n)}$ into this formula simply by adding and subtracting the term $m(x+h)n(x)$.

$$\text{ARC}_h^{(f)}(x) = \frac{m(x+h)n(x+h) - m(x)n(x)}{h}$$

$$= \frac{m(x+h)n(x+h) - m(x+h)n(x) + m(x+h)n(x) - m(x)n(x)}{h}$$

$$= m(x+h)\left[\frac{n(x+h)-n(x)}{h}\right] + n(x)\left[\frac{m(x+h)-m(x)}{h}\right]$$

$$= m(x+h)\cdot \text{ARC}_h^{(n)}(x) + n(x)\cdot \text{ARC}_h^{(m)}(x)$$

The last formula suggests the correct rule relating the derivative of f to the derivatives of its factors.[1]

Product Rule for Derivatives

If $y = m(x)$ and $y = n(x)$ are differentiable functions, then the function $y = f(x) = m(x)\cdot n(x)$ is also differentiable, and

$$f'(x) = m(x)\cdot n'(x) + n(x)\cdot m'(x) = m(x)\cdot\frac{dn}{dx}(x) + n(x)\cdot\frac{dm}{dx}(x)$$

Example 1. Use the specific derivative formulas and the product rule to differentiate the function

$$f(x) = 3\cos(x)\cdot\ln(x)$$

Solution. When using derivative rules to differentiate a function, the general strategy is always the same. You apply general rules to break up the formula for the function into combinations of derivatives of progressively simpler formulas, until you are faced with derivatives that are *exact matches* to the specific derivative formulas.

$$\frac{df}{dx}(x) = \frac{d}{dx}\Big[\overbrace{\underbrace{3\cos(x)}_{m(x)}\cdot\underbrace{\ln(x)}_{n(x)}}^{\text{The formula is a product of two functions}}\Big]$$

$$= 3\cos(x)\cdot\frac{d}{dx}[\ln(x)] + \ln(x)\cdot\frac{d}{dx}[3\cos(x)] \qquad \textit{Apply Product Rule}$$

$$= 3\cos(x)\cdot\frac{d}{dx}[\ln(x)] + \ln(x)\cdot 3\frac{d}{dx}[\cos(x)] \qquad \textit{Apply Constant Multiple Rule}$$

$$= 3\cos(x)\cdot\frac{1}{x} + \ln(x)\cdot 3(-\sin(x)) \qquad \textit{Apply Specific Formulas}$$

$$= 3\left[\frac{\cos(x)}{x} - \ln(x)\cdot\sin(x)\right] \qquad \textit{Make easy simplifications}$$

[1] There are technical details involved in working with the limiting process which we will leave to a course on the theory of calculus.

Problem 2. Consider the function $a = h(t) = t^3 \cdot \ln(t)$ in the interval $0 < x < \infty$. What is the slope of the tangent line to the graph of h at the point $(1,0)$?

Problem 3. Use the Product Rule and Sum Rule to show that $y = F(x) = x \cdot \ln(x) - x$ is one antiderivative for the function $r = f(x) = \ln(x)$ in the interval $0 < x < \infty$.

Problem 4. Consider the function $y = f(x) = x^4$. Use the Product Rule and the fact that $f(x) = x^3 \cdot x$ to construct the derivative formula for f. Simplify your answer as much as possible.

Problem 5. Consider the function $y = f(x) = x^5$.

Part (a). Make a conjecture about what the formula for the derivative $r = f'(x)$ should be.

Part (c). Using Problem 4 as a model, test your conjecture using the Product Rule.

Problem 6. Consider the function $y = f(x) = \frac{1}{x^2}$. Use the fact that $f(x) = \frac{1}{x} \cdot \frac{1}{x}$ and the Product Rule to determine the formula for $r = f'(x)$. Simplify your answer as much as possible.

Problem 7. Consider the function $y = f(x) = \frac{1}{x^3}$. Use the fact that $f(x) = \frac{1}{x^2} \cdot \frac{1}{x}$ and the Product Rule to determine the formula for $r = f'(x)$. Simplify your answer as much as possible.

Integer Power Formula
If $y = f(x) = x^n$ for any nonzero integer n then $f'(x) = nx^{n-1}$.

Problem 8. What is the formula for $r = f'(x)$ if we know $y = f(x) = x^{-6}\cos(x)$?

Homework.

Use the general rules and specific derivative formulas to differentiate the following functions.

(1) $f(x) = \frac{\sqrt{x}\cos(x)}{5}$

(2) $g(u) = \cos(u)\sin(u)$

(3) $h(v) = \frac{\sin(v)\cdot e^v}{\pi}$

(4) $f(x) = 4x^7 - 3x^{-10} + 8$

(5) $g(u) = u^5 \cdot e^u$

(6) $h(v) = v\cos(v) - 4v^8$

(7) $f(x) = x^2 \cdot \log_5(x)$

(8) $g(u) = \left(\frac{1}{u^3}\right)\cos(u) - 2\log_2(u)$

(9) $h(v) = \frac{e^v}{\sqrt{3}\cdot v^5} - \frac{1}{v^6} + 9$

Problem 10. Consider the function $y = f(x) = x^{3/2}$.

Part (a). Use the laws of exponents to show that $f(x) = x \cdot \sqrt{x}$.

Part (b). Use the Product Rule to differentiate the function f, then simplify your answer as much as possible.

Problem 11. Consider the function $y = f(x) = x^{5/2}$.

Part (a). Use the laws of exponents to show that $f(x) = x^2 \cdot \sqrt{x}$.

Part (b). Use the Product Rule to differentiate the function f then simplify your answer as much as possible.

Problem 12. Consider the function $y = f(x) = x^{7/2}$.

Part (a). Use the laws of exponents to show that $f(x) = \sqrt{x} \cdot x^3$.

Part (b). Use the Product Rule to differentiate the function f, then simplify your answer as much as possible.

Problem 13. Based on your work in Problems 10 – 12, what would you predict the derivative formula should be for the function $y = f(x) = x^{9/2}$? Use the method in Problems 10 – 12 to verify your prediction.

Problem 14. Consider the function $y = f(x) = e^{2x}$.

Part (a). Use the laws of exponents to show that $f(x) = e^x \cdot e^x$.

Part (b). Use the Product Rule to differentiate the function f.

Problem 15. How could you use the Product Rule to differentiate the function $y = f(x) = xe^x \sin(x)$?

Problem 16. Use the Product Rule and Sum Rule to show that $y = F(x) = (x-1) \cdot e^x$ is one antiderivative for the function $r = f(x) = x \cdot e^x$ in the interval $-\infty < x < \infty$.

Problem 17. Show that, when n is an integer and $n \neq -1$, the function $y = F(x) = \frac{1}{n+1} x^{n+1}$ is one antiderivative for the function $r = f(x) = x^n$.

Problem 18. Show that $y = F(x) = x^2 \sin(x) + 2x\cos(x) - 2\sin(x)$ is one antiderivative for the function $r = f(x) = x^2 \cos(x)$.

Construct one antiderivative for each of the following functions.

(19) $f(x) = x^2 + 2\sin(x)$	(20) $g(y) = y^{-3} + e^y - \sqrt{5}$	(21) $h(p) = \dfrac{4}{p^2} - \pi \ln(p)$

Answers to the Homework.

(1) $f'(x) = \frac{1}{5}\left(\frac{\cos(x)}{2\sqrt{x}} - \sqrt{x}\sin(x)\right)$

(2) $g'(u) = [\cos(x)]^2 - [\sin(x)]^2$

(3) $h'(v) = \frac{e^v}{\pi}(\cos(v) + \sin(v))$

(4) $f'(x) = 28x^6 + 30x^{-11}$

(5) $g'(u) = e^u u^4 (5 + u)$

(6) $h'(v) = \cos(v) - v\sin(v) - 32v^7$

(7) $f'(x) = 2x \cdot \log_5(x) + \frac{x}{\ln(5)}$

(8) $g'(u) = -\left(\frac{3}{u^4}\right)\cos(u) - \left(\frac{1}{u^3}\right)\sin(u) - \frac{2}{\ln(2)}$

(9) $h'(v) = \frac{e^v}{\sqrt{3}\cdot v^5} - \frac{5\cdot e^v}{\sqrt{3}\cdot v^6} + \frac{6}{v^7}$

Problem 10. First, observe that

$$x^{3/2} = x^{1+1/2} = x^1 \cdot x^{1/2} = x \cdot \sqrt{x}$$

$$\frac{d}{dx}\left[x^{3/2}\right] = \frac{d}{dx}[x] \cdot \sqrt{x} + x \cdot \frac{d}{dx}\left[\sqrt{x}\right] = (1) \cdot \sqrt{x} + x \cdot \frac{1}{2\sqrt{x}}$$

We can simplify the last expression quite a bit by rewriting it as a single fraction. Observe

$$\frac{d}{dx}\left[x^{3/2}\right] = \frac{2x + x}{2\sqrt{x}} = \frac{3x}{2\sqrt{x}} = \frac{3}{2}\sqrt{x} = \frac{3}{2} \cdot x^{1/2}$$

Problem 11. First, observe that

$$x^{5/2} = x^{2+1/2} = x^2 \cdot \sqrt{x}$$

$$\frac{d}{dx}\left[x^{5/2}\right] = \frac{d}{dx}[x^2] \cdot \sqrt{x} + x^2 \cdot \frac{d}{dx}\left[\sqrt{x}\right] = 2x \cdot \sqrt{x} + x^2 \cdot \frac{1}{2\sqrt{x}}$$

We can simplify the last expression quite a bit by rewriting it as a single fraction. Observe

$$\frac{d}{dx}\left[x^{5/2}\right] = \frac{\left(2x\sqrt{x}\right) \cdot 2\sqrt{x} + x^2}{2\sqrt{x}} = \frac{4x^2 + x^2}{2\sqrt{x}} = \frac{5x^2}{2\sqrt{x}} = \frac{5}{2} \cdot x^{3/2}$$

Problem 12. First, observe that

$$x^{7/2} = x^{3+1/2} = x^3 \cdot \sqrt{x}$$

$$\frac{d}{dx}\left[x^{7/2}\right] = \frac{d}{dx}[x^3] \cdot \sqrt{x} + x^3 \cdot \frac{d}{dx}\left[\sqrt{x}\right] = 3x^2 \cdot \sqrt{x} + x^3 \cdot \frac{1}{2\sqrt{x}}$$

We can simplify the last expression quite a bit by rewriting it as a single fraction. Observe

$$\frac{d}{dx}\left[x^{7/2}\right] = \frac{\left(3x^2\sqrt{x}\right) \cdot 2\sqrt{x} + x^3}{2\sqrt{x}} = \frac{6x^3 + x^3}{2\sqrt{x}} = \frac{7x^3}{2\sqrt{x}} = \frac{7}{2} \cdot x^{5/2}$$

Problem 13. First, observe that

$$x^{9/2} = x^{4+1/2} = x^4 \cdot \sqrt{x}$$

$$\frac{d}{dx}\left[x^{9/2}\right] = \frac{d}{dx}[x^4] \cdot \sqrt{x} + x^4 \cdot \frac{d}{dx}\left[\sqrt{x}\right] = 4x^3 \cdot \sqrt{x} + x^4 \cdot \frac{1}{2\sqrt{x}}$$

We can simplify the last expression quite a bit by rewriting it as a single fraction. Observe

$$\frac{d}{dx}\left[x^{9/2}\right] = \frac{\left(4x^3\sqrt{x}\right) \cdot 2\sqrt{x} + x^4}{2\sqrt{x}} = \frac{8x^4 + x^4}{2\sqrt{x}} = \frac{9x^4}{2\sqrt{x}} = \frac{9}{2} \cdot x^{7/2}$$

Problem 14. First, observe that

$$e^{2x} = e^{x+x} = e^x \cdot e^x$$

$$\frac{d}{dx}[e^{2x}] = e^x \cdot \frac{d}{dx}[e^x] + e^x \cdot \frac{d}{dx}[e^x] = e^x \cdot e^x + e^x \cdot e^x = 2e^{2x}$$

Problem 15. We can apply the Product Rule twice. Observe

$$\begin{aligned}\frac{d}{dx}[xe^x\sin(x)] &= xe^x\cdot\frac{d}{dx}[\sin(x)]+\frac{d}{dx}[xe^x]\cdot\sin(x)\\ &= xe^x\cdot\frac{d}{dx}[\sin(x)]+\left(x\cdot\frac{d}{dx}[e^x]+\frac{d}{dx}[x]\cdot e^x\right)\cdot\sin(x)\\ &= xe^x(\cos(x)+\sin(x))+\ e^x\sin(x)\end{aligned}$$

Problem 16. Observe that

$$\begin{aligned}\frac{d}{dx}[(x-1)e^x] &= (x-1)\cdot\frac{d}{dx}[e^x]+\frac{d}{dx}[x-1]\cdot e^x\\ &= (x-1)e^x+(1-0)e^x\\ &= xe^x-e^x+e^x\\ &= xe^x\end{aligned}$$

Problem 17. Observe that

$$\frac{d}{dx}\left[\frac{1}{n+1}x^{n+1}\right]=\frac{1}{n+1}\cdot\frac{d}{dx}[x^{n+1}]=\frac{1}{n+1}(n+1)x^{n+1-1}=x^n$$

Problem 18. Observe that

$$\begin{aligned}\frac{d}{dx}[x^2\sin(x)+2x\cos(x)-2\sin(x)] &= \frac{d}{dx}[x^2\sin(x)]+\frac{d}{dx}[2x\cos(x)]-\frac{d}{dx}[2\sin(x)]\\ &= \frac{d}{dx}[x^2\sin(x)]+2\frac{d}{dx}[x\cos(x)]-2\frac{d}{dx}[\sin(x)]\\ &= x^2\cdot\frac{d}{dx}[\sin(x)]+\frac{d}{dx}[x^2]\cdot\sin(x)+2\left(x\cdot\frac{d}{dx}[\cos(x)]+\frac{d}{dx}[x]\cdot\cos(x)\right)-2\frac{d}{dx}[\sin(x)]\\ &= x^2\cos(x)+2x\sin(x)-2x\sin(x)+2\cos(x)-2\cos(x)\\ &= x^2\cos(x)\end{aligned}$$

(19) $F(x)=\dfrac{1}{3}x^3-2\cos(x)$

(20) $G(y)=-\dfrac{1}{2y^2}+e^y-\sqrt{5}\,y$

(21) $H(p)=-\dfrac{4}{p}-\pi p\ln(p)+\pi p$