Suppose that $y = f(x)$ is a differentiable function, and consider the reciprocal function

$$y = R(x) = \frac{1}{f(x)}$$

Observe that the average rate of change functions for the function R are given by the formula

$$\mathrm{ARC}_h(x) = \frac{R(x+h) - R(x)}{h} = \left(\frac{1}{h}\right)\left[\frac{1}{f(x+h)} - \frac{1}{f(x)}\right] = \left(\frac{1}{h}\right)\left[\frac{f(x) - f(x+h)}{f(x+h)\cdot f(x)}\right]$$

Now, since $f(x) - f(x+h) = -[f(x+h) - f(x)]$, we know

$$\mathrm{ARC}_h(x) = \left(\frac{1}{h}\right)\left[\frac{-[f(x+h) - f(x)]}{f(x+h)\cdot f(x)}\right] = \left[\frac{-1}{f(x+h)\cdot f(x)}\right]\cdot\left[\frac{f(x+h) - f(x)}{h}\right] = \frac{-1}{f(x+h)\cdot f(x)}\cdot ARC_h^{(f)}(x)$$

Since we know that the function f is continuous, we also know that

$$\lim_{h\to 0} f(x+h) = f(x)$$

Therefore, we also know

$$\begin{aligned} R'(x) &= \lim_{h\to 0} \mathrm{ARC}_h(x) \\ &= -\left(\lim_{h\to 0}\frac{1}{f(x+h)\cdot f(x)}\right)\cdot\left(\lim_{h\to 0} ARC_h^{(f)}(x)\right) = -\left(\frac{1}{f(x)\cdot f(x)}\right)\cdot f'(x) \end{aligned}$$

Reciprocal Rule for Derivatives

Suppose that $y = f(x)$ is a differentiable function. As long as $f(x) \neq 0$, the reciprocal function $y = R(x) = \frac{1}{f(x)}$ is also differentiable, and

$$R'(x) = -\frac{f'(x)}{f(x)\cdot f(x)} = -\frac{1}{f(x)\cdot f(x)}\cdot\frac{df}{dx}(x)$$

The *secant* and *tangent* functions are defined by the formulas

$$H(\theta) = \sec(\theta) = \frac{1}{\cos(\theta)} \qquad\qquad G(\theta) = \tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)}$$

Problem 1. Use the Reciprocal Rule to show that

$$\frac{d}{d\theta}[\sec(\theta)] = \sec(\theta)\tan(\theta)$$

Problem 2. Use the Product Rule and the Reciprocal Rule to construct the derivative function for

$$y = g(x) = \left(\frac{1}{x-1}\right) \cdot (x-2)$$

Write your answer as a single fraction.

The process of applying the Product Rule and Reciprocal Rule to differentiate a quotient of two functions can be summarized in a general rule called the *Quotient Rule* for derivatives. The Quotient Rule provides a shortcut for differentiating any quotient of differentiable functions; however, you can always use the Product Rule and Reciprocal Rule together to accomplish the same thing.

Quotient Rule for Derivatives

Suppose that $y = U(x)$ and $y = L(x)$ are differentiable functions. As long as $L(x) \neq 0$, the function

$$y = f(x) = \frac{U(x)}{L(x)}$$

is also differentiable, and

$$f'(x) = \frac{L(x) \cdot U'(x) - U(x) \cdot L'(x)}{L(x) \cdot L(x)} = \frac{1}{L(x) \cdot L(x)}\left(L(x)\frac{dU}{dx}(x) - U(x)\frac{dL}{dx}(x)\right)$$

To see why the Quotient Rule is valid, observe

$$\frac{df}{dx} = \frac{d}{dx}\left[\frac{1}{L(x)} \cdot U(x)\right]$$

$$= \frac{1}{L(x)} \cdot \frac{d}{dx}[U(x)] + \frac{d}{dx}\left[\frac{1}{L(x)}\right] \cdot U(x)$$ Apply the Product Rule

$$= \frac{U'(x)}{L(x)} - \left(\frac{L'(x)}{L(x) \cdot L(x)}\right) \cdot U(x)$$ Apply the Reciprocal Rule

$$= \frac{U'(x) \cdot L(x) - L'(x) \cdot U(x)}{L(x) \cdot L(x)}$$

Example 1. Construct the formula for the derivative function for the function

$$p = q(r) = 4r - \frac{r}{1+r^2}$$

Solution. We will need to apply several general rules to construct this formula. Observe

$$\frac{dp}{dr} = \frac{d}{dr}\Bigg[\overbrace{4r - \frac{r}{1+r^2}}^{\text{This formula is a sum of two functions}}\Bigg] \quad \text{This formula is a quotient of two functions}$$

$$= 4\frac{d}{dr}[r] \quad - \quad \frac{d}{dr}\Bigg[\overbrace{\frac{r}{1+r^2}}\Bigg] \quad \textit{Apply Sum and Constant Multiple Rules}$$

$$= 4\frac{d}{dr}[r] - \left[\frac{1}{(1+r^2)^2}\right]\left[(1+r^2)\frac{d}{dr}[r] - r\frac{d}{dr}\overbrace{[1+r^2]}^{\text{This formula is a sum of two functions}}\right] \quad \textit{Apply Quotient Rule}$$

$$= 4\frac{d}{dr}[r] - \left[\frac{1}{(1+r^2)^2}\right]\left[(1+r^2)\frac{d}{dr}[r] - r\left(\frac{d}{dr}[1] + \frac{d}{dr}[r^2]\right)\right] \quad \textit{Apply Sum Rule}$$

$$= 4(1) - \left[\frac{1}{(1+r^2)^2}\right][(1+r^2)(1) - r(0+2r)] \quad \textit{Apply specific derivative formulas}$$

$$= 4 - \frac{1-r^2}{(1+r^2)^2}$$

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Problem 3. Use Quotient Rule to construct the derivative formula for the function

$$y = g(x) = \frac{\sin(x)}{1+x^2}$$

Problem 4. Use the Quotient Rule and the *Pythagorean Identity* $[\sin(\theta)]^2 + [\cos(\theta)]^2 = 1$ to show that

$$\frac{d}{d\theta}[\tan(\theta)] = [\sec(\theta)]^2$$

Homework.

Use the Product, Reciprocal, or Quotient Rules to construct the derivative functions for the following functions.

(1) $f(x) = (3x^{-2} - 5)\tan(x)$

(2) $g(a) = \frac{1}{3+4x^4}$

(3) $h(w) = e^w \sec(w)$

(4) $f(x) = \frac{\sin(x)}{x-3}$

(5) $g(a) = \frac{1-a}{1+a}$

(6) $h(w) = \frac{1}{w^2-4w}$

(7) $f(x) = \sqrt{x}(x^{-3} + 3x - 1)$

(8) $g(a) = \frac{3e^a \cdot \ln(a)}{2}$

(9) $h(w) = \frac{e}{we^w}$

Problem 10. Construct the formula for the line tangent to the graph of the function

$$y = f(x) = \frac{x}{2} - \frac{x}{x+1}$$

at the point $(1, f(1))$.

Problem 11. Identify one antiderivative for the function $r = f(x) = \sec(x) \cdot (2\sec(x) + 3\tan(x))$.

Answers to the Homework.

(1) $f'(x) = (3x^{-2} - 5)[\sec(x)]^2 - 6x^{-3}\tan(x)$

(2) $g'(a) = -\frac{16x^3}{(3+4x^4)^2}$

(3) $h'(w) = e^w \sec(w)[1 + \tan(w)]$

(4) $f'(x) = \frac{(x-3)\cos(x) - \sin(x)}{(x-3)^2}$

(5) $g'(a) = -\frac{2}{(1+a)^2}$

(6) $h'(w) = \frac{4-2w}{(w^2-4w)^2}$

(7) $f'(x) = \frac{x^{-3}+3x-1}{2\sqrt{x}} + 3\sqrt{x}(1 - x^{-4})$

(8) $g'(a) = \frac{3e^a}{2}\left(\ln(a) + \frac{1}{a}\right)$

(9) $h'(w) = -\frac{e}{w^2 e^w}(1 + w)$

Problem 10. First, note that $f(1) = 0$. Now, observe that

$$f'(x) = \frac{1}{2} - \frac{1}{(x+1)^2} \qquad \Rightarrow \qquad f'(1) = \frac{1}{4}$$

The formula for the tangent line will therefore be $y = T(x) = \frac{1}{4}(x - 1)$.

Problem 11. First, observe that $f(x) = 2[\sec(x)]^2 + 3\sec(x)\tan(x)$. Consequently, one antiderivative would be $y = F(x) = 2\tan(x) + 3\sec(x)$.