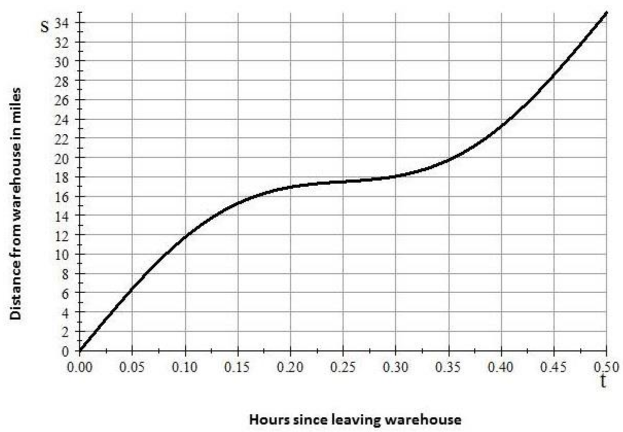
In this investigation, we will introduce what is arguably the most important insight that calculus gives us regarding covariation. It may not seem like much at first, but this result underlies virtually every technique used in Calculus I. Let's start with a problem.

The farm-to-market trucking company Haulin' Oats LLC is very proud of its safety record and has a strict no-tolerance policy regarding speed limits. The company requires all of its employees to keep their trucking speeds under 65 mph. The company uses a satellite tracking device on every truck that records the distance traveled from the pickup site as well as the truck velocity.

Sathya picked up a load of barley at a warehouse in Kansas City and delivered it to the central warehouse west of Topeka exactly half an hour later as recorded by the company foreman. His tracking device malfunctioned and did not record his velocity; however, it did manage to record his distance from the warehouse in Kansas City. The diagram below shows Sathya's distance s (measured in miles) from the warehouse in Kansas City as a function of the number t of hours since he left the warehouse.



Barbara, the human resources manager for Haulin' Oats, looked at the graph and fired Sathya on the spot. Sathya appealed the dismissal, claiming that, while his average velocity on the time interval $0 \leq t \leq 0.5$ hours did exceed the maximum allowed velocity, his *instantaneous velocity* never did.

Problem 1. What was Sathya's average velocity on the time interval $0 \leq t \leq 0.5$ hours?

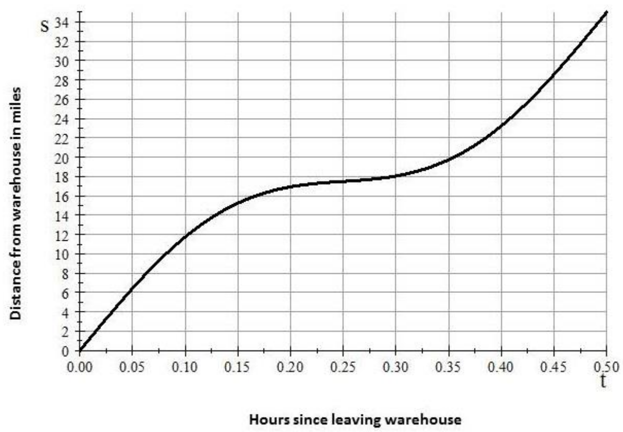
Problem 2. Do you think Sathya's instantaneous velocity ever exceeded the 65 mph requirement? Justify your thinking using the graph.

Problem 3. Let $s = f(t)$ be the function that gives the values of s in terms of the values of t.

Part (a). On the graph provided, draw the secant line for the function $s = f(t)$ on the time interval $0 \leq t \leq 0.5$ hours.

Part (b). Are there any time values where the tangent line to the graph of f is parallel to this secant line?

Part (c). Explain how your answer to Part (b) allows you to decide whether Sathya should be reinstated.



> ***Mean Value Theorem***
>
> Suppose that $y = f(x)$ is a function that is continuous on the closed input interval $a \leq x \leq b$ and differentiable on the open input interval $a < x < b$. There will always be at least one input value $x = c$ where the tangent line to the graph of the function f is parallel to the secant line passing through the points $(a, f(a))$ and $(b, f(b))$.

Problem 4. Sathya countered by claiming that the tracking device's data could not be correct, citing as evidence the graph on the input interval $0 \leq t \leq 0.10$ hours and the fact that his truck is incapable of velocities greater than 100 mph. Do you agree with Sathya? Why or why not?

Problem 5. Barbara stated that the actual data is irrelevant --- Sathya traveled 35 miles in thirty minutes. His instantaneous velocity *must* have been at least 70 mph at some time. Does the Mean Value Theorem support Barbara's conclusion? Explain.

First Corollary of the Mean Value Theorem

Suppose that $y = f(x)$ is a function that is differentiable in the open interval $a < x < b$. If $f'(x) = 0$ for all input values in the interval, then f is a constant output function in this interval.

To prove that this result is true, we need to show that $f(u) = f(v)$ for every pair $x = u$, $x = v$ in the interval. Since the function f is differentiable in this interval, we know that f is continuous in this interval. (See Investigation 8.) In particular, $f(u)$ and $f(v)$ are defined for *any* input values $x = u$ and $x = v$ that we choose. Let's assume that $u < v$.

Now, by assumption, f is continuous on the interval $[u, v]$ since this is a subinterval of $a < x < b$; and of course, f is differentiable on the open interval $u < x < v$ by assumption. Therefore, the Mean Value Theorem tells us there is an input value $x = c$ in the interval $u < x < v$ such that

$$\frac{f(u) - f(v)}{u - v} = f'(c)$$

By assumption, $f'(c) = 0$, so we must conclude that $f(u) = f(v)$ as desired.

Second Corollary of the Mean Value Theorem

Suppose that $r = f(x)$ is a function in an open interval $a < x < b$. If $y = F(x)$ and $y = G(x)$ are both antiderivatives for f, then $H(x) = F(x) - G(x)$ is a constant output function.

Problem 6. To prove that this result is true, let $H(x) = F(x) - G(x)$ and use the Sum Rule to show $H'(x) = 0$. Why does this imply that H is a constant output function?

If $y = F(x)$ is one antiderivative for a function $r = f(x)$, then we have known for some time that we can create additional antiderivatives for f by letting $G(x) = F(x) + C$ for any fixed real number C. The Second Corollary of the Mean Value Theorem tells us that such functions are the *only* antiderivatives for f. This fact has profound consequences, as we shall see in later investigations.

Problem 7. Consider the function $f(x) = \sin(x)\cos(x)$, along with the functions

$$F(x) = \frac{\sin(x) \cdot \sin(x)}{2} \qquad\qquad G(x) = -\frac{\cos(x) \cdot \cos(x)}{2}$$

Part (a). By differentiating the functions F and G, show that *both* serve as antiderivatives for the function f.

Part (b). Since both of the functions F and G serve as antiderivatives for the function f, there must exist a constant C such that $F(x) = G(x) + C$. What is the value of this constant?

Homework.

Problem 1. Consider the function $f(x) = 2x - \sqrt{x}$.

Part (a). Show that, for any constant C, the function $F(x) = x^2 - \frac{2}{3}x\sqrt{x} + C$ is an antiderivative for f.

Part (b). If we want $F(1) = 2$, what is the value of C?

Part (c). If we want $F(1) = -2$, what is the value of C?

Problem 2. Consider the function $f(x) = x^{-1}\cos(x) - x^{-2}\sin(x)$.

Part (a). Show that, for any constant C, the function $F(x) = x^{-1}\sin(x)$ is an antiderivative for f.

Part (b). If we want $F(\pi) = -1$, what is the value of C?

Part (c). If we want $F(2) = 5$, what is the approximate value of C?

The set of *all* antiderivatives for a given function $y = f(x)$ is called the *antiderivative family* for the function f. The Mean Value Theorem tells us that once we know one antiderivative for f, we know them all --- the others are simply vertical translations of the one we know. In calculus, it is customary to use the special notation

$$\int f(x)\,dx$$

to mean "construct an *arbitrary* antiderivative for the function $y = f(x)$."[1] For example, we would write

$$\int \left(2x - 3\sqrt{x}\right) dx = x^2 - 2x\sqrt{x} + C$$

Construct formulas for the following arbitrary antiderivatives.

(3) $\int \left(a - \frac{1}{\sqrt{a}} + 5\right) da$	(4) $\int \sqrt{3}\ dt$	(5) $\int (\sin(q) - 4\cos(q))\,dq$
(6) $\int \left(2\sec^2(a) + \sqrt{5}\ln(a)\right)\ da$	(7) $\int \left(\frac{e}{t^2} - \frac{t^3}{\sqrt{6}}\right)\ dt$	(8) $\int (1 + 2\pi\sec(q)\tan(q))\ dq$

[1] We will explain this rather strange notation in Investigation 21.

Special Derivative and Antiderivative Formulas We Have Developed So Far

1. Since $\frac{d}{dx}[Kx] = K$ for any constant K, we know $\int K\,dx = Kx + C$.
2. Since $\frac{d}{dx}[\sin(x)] = \cos(x)$, we know $\int \cos(x)\,dx = \sin(x) + C$.
3. Since $\frac{d}{dx}[\cos(x)] = -\sin(x)$, we know $\int \sin(x)\,dx = -\cos(x) + C$.
4. Since $\frac{d}{dx}[\tan(x)] = \sec^2(x)$, we know $\int \sec^2(x)\,dx = \tan(x) + C$.
5. Since $\frac{d}{dx}[\sec(x)] = \sec(x)\tan(x)$, we know $\int \sec(x)\tan(x)\,dx = \sec(x) + C$.
6. Since $\frac{d}{dx}[\ln(x)] = \frac{1}{x}$, we know $\int \frac{1}{x}\,dx = \ln(x) + C$ --- but only when $x > 0$.
7. Since $\frac{d}{dx}[x\ln(x) - x] = \ln(x)$, we know $\int \ln(x)\,dx = x\ln(x) - x + C$.
8. Since $\frac{d}{dx}[x^{n+1}] = (\mathrm{n}+1)x^n$, when $n \neq -1$ is an integer, we know $\int x^n\,dx = \frac{1}{n+1}x^{n+1} + C$.
9. Since $\frac{d}{dx}[\sqrt{x}] = \frac{1}{2\sqrt{x}}$, we know $\int \frac{1}{\sqrt{x}}\,dx = 2\sqrt{x} + C$.
10. Since $\frac{d}{dx}[e^x] = e^x$, we know $\int e^x\,dx = e^x + C$.

Answers to the Homework.

Problem 1.

Part (a). Observe

$$\frac{d}{dx}\left[x^2 - \frac{2}{3}x\sqrt{x} + C\right] = \frac{d}{dx}[x^2] - \frac{2}{3}\cdot\frac{d}{dx}\left[x\sqrt{x}\right] + \frac{d}{dx}[C]$$

$$= \frac{d}{dx}[x^2] - \frac{2}{3}\cdot\left(x\cdot\frac{d}{dx}\left[\sqrt{x}\right] + \sqrt{x}\cdot\frac{d}{dx}[x]\right) + \frac{d}{dx}[C]$$

$$= 2x - \frac{2}{3}\cdot\left(\frac{x}{2\sqrt{x}} + \sqrt{x}\right)$$

$$= 2x - \frac{2}{3}\cdot\left(\frac{3x}{2\sqrt{x}}\right)$$

$$= 2x - \sqrt{x}$$

Part (b). We know $2 = F(1) = 1^2 - \frac{2}{3}(1)\sqrt{1} + C$, so $C = \frac{5}{3}$.

Part (c). We know $-2 = F(1) = 1^2 - \frac{2}{3}(1)\sqrt{1} + C$, so $C = -\frac{7}{3}$.

Problem 2.

Part (a). Observe

$$\frac{d}{dx}[x^{-1}\sin(x)\,] = x^{-1}\cdot\frac{d}{dx}[\sin(x)] + \sin(x)\cdot\frac{d}{dx}[x^{-1}] = x^{-1}\cos(x) - x^{-2}\sin(x)$$

Part (b). We know $-1 = F(\pi) = \frac{\cos(\pi)}{\pi} - \frac{\sin(\pi)}{\pi^2} + C$, so $C = \frac{1}{\pi} - 1$.

Part (c). We know $5 = F(2) = \frac{\cos(2)}{2} - \frac{\sin(2)}{4} + C$, so $C = 5 + \frac{\sin(2)}{4} - \frac{\cos(2)}{2} \approx 5.435$.

(3) $\int \left(a - \frac{1}{\sqrt{a}} + 5\right) da = \frac{1}{2}a^2 - 2\sqrt{a} + 5a + C$

(4) $\int \sqrt{3}\ dt = \sqrt{3}\,t + C$

(5) $\int (\sin(q) - 4\cos(q))\,dq = -\cos(q) - 4\sin(q) + C$

(6) $\int \left(2\sec^2(a) + \sqrt{5}\ln(a)\right)\ da = 2\tan(a) + \sqrt{5}\,a\ln(a) - \sqrt{5}\,a + C$

(7) $\int \left(\frac{e}{t^2} - \frac{t^3}{\sqrt{6}}\right) dt = -\frac{e}{t} - \frac{t^4}{4\sqrt{6}} + C$

(8) $\int (1 + 2\pi\sec(q)\tan(q))\ dq = q + 2\pi\sec(q) + C$