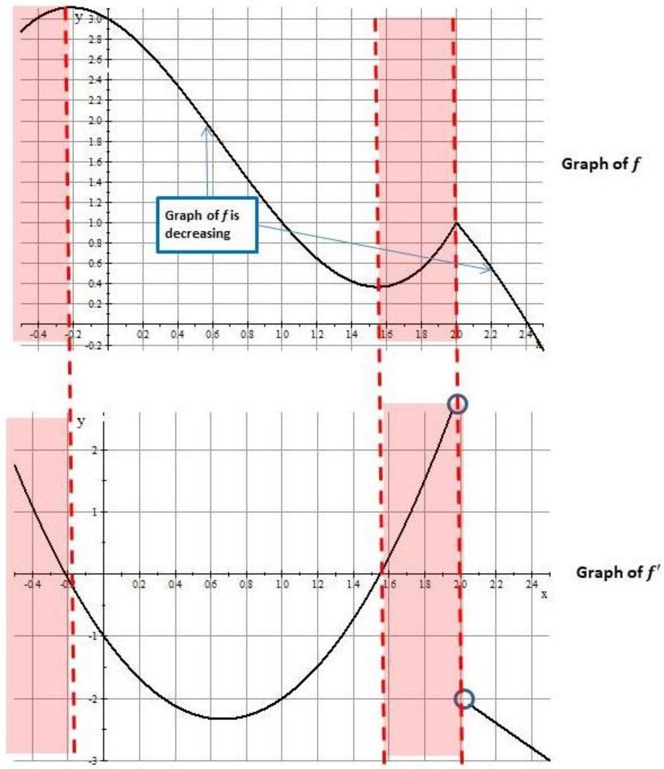
The diagram below shows the graph and derivative function for the piecewise-defined function

$$y = f(x) = \begin{cases} 3 + x^3 - 2x^2 - x & \text{if} \quad x < 2 \\ 2 - (x-1)^2 & \text{if} \quad 2 \le x \end{cases}$$



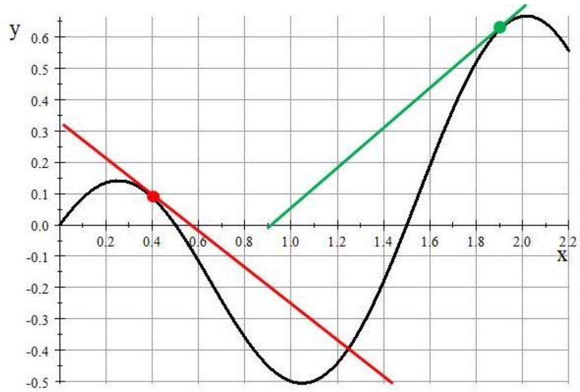
Problem 1. Consider the diagram above.

Part (a). Why is there a jump in the graph of the derivative function at the input value $x = 2$?

Part (b). According to the diagram, $f'(1.0) = -2.0$. What are some things this tells us about the function f at the input value $x = 1.0$?

Part (c). According to the diagram, $f'(1.8) \approx 1.5$. How was this value obtained from the function f?

Problem 3. The trace below shows the graph of a function $y = f(x)$, along with two tangent lines to the graph.

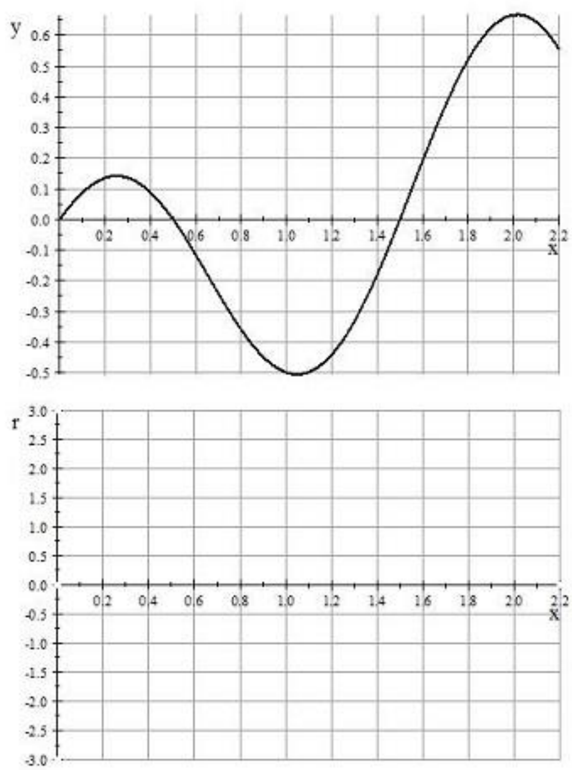


Part (a). Use these tangent lines to estimate the value of $f'(0.4)$ and $f'(1.9)$. What is your strategy?

Part (b). Using tangent lines to estimate outputs, complete the following table.

Input Value	0.0	0.2	0.6	0.8	1.0	1.2	1.6	2.0	2.2
Approximate Value of $f'(x)$									

Problem 4. Use the output values you estimated in Problem 3 to sketch an approximation for the graph of the derivative function for f.



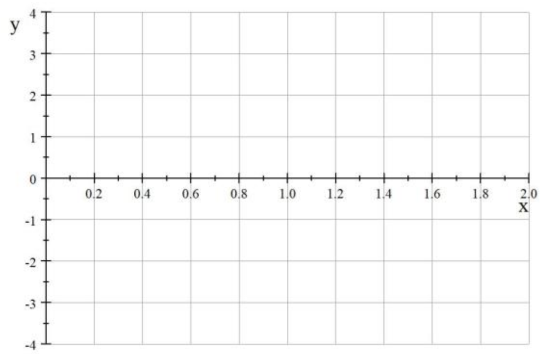
Third Corollary to the Mean Value Theorem

Let $y = f(x)$ be a differentiable function.

- If $f'(x) > 0$ for all input values x in some open interval $a < x < b$, then the graph of the function f is increasing on this interval.
- If $f'(x) < 0$ for all input values x in some open interval $a < x < b$, then the graph of the function f is decreasing on this interval.

Problem 5. Let's investigate *why* the Mean Value Theorem helps us to establish the truth of the first statement. Assume that $y = f(x)$ is any function that is differentiable on the input interval $0 \le x \le 2$, and suppose $r = f'(x)$ has positive output on this interval.

Choose any two input values $x = u$ and $x = v$ you like on the input axis below --- only make sure $u < v$.



Part (a). We have assumed that f is differentiable on the input interval $0 \le x \le 2$. Explain why this guarantees that $f(u)$ and $f(v)$ exist.

Part (b). Choose any value for $f(u)$ and $f(v)$ you like --- so long as both choices are between -4 and 4. Draw the points $(u, f(u))$ and $(v, f(v))$ on the grid above.

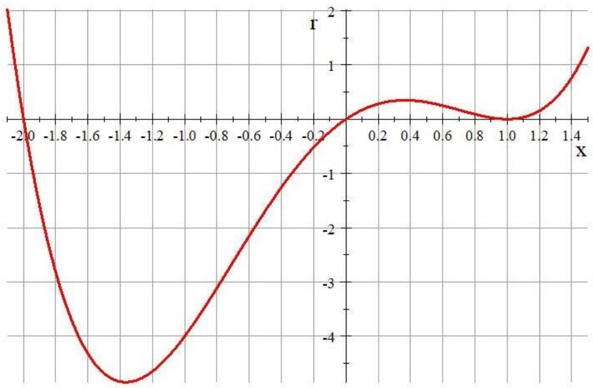
Part (c). You don't know exactly what the graph of f looks like, but assume it passes through the two points you drew. Explain why we know the graph of f *cannot* have any jumps, holes, or vertical asymptotes on your input interval $u \le x \le v$. (In other words, the function f *must* be continuous on your chosen input interval.)

Part (d). Draw the straight line segment that passes through the two points $(u, f(u))$ and $(v, f(v))$. What does the slope of this line segment represent?

Again, we don't know what the graph of f looks like, but the Mean Value Theorem *guarantees* there is some input value $x = a$ in your input interval $u \leq x \leq v$ where the slope of the tangent line to the graph of f is the same as the slope of your line segment.

Part (e). Now, here is the key --- we are *assuming* that $f'(a) > 0$. That is, we are *assuming* the slope of the tangent line to the graph of f at $(a, f(a))$ is positive. Is it possible for the slope of this tangent line to be positive *and* be equal to the slope of your line segment? If not, what would you need to change about your choice of $f(v)$ in order to make it possible?

Problem 6. The diagram below shows the graph of the *derivative function* for a function $y = f(x)$.



Graph of f'

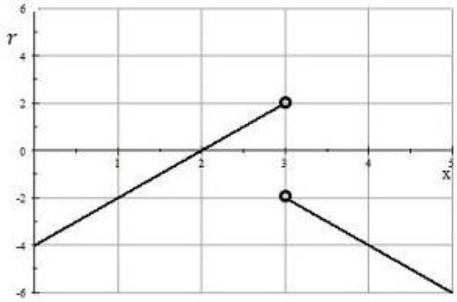
Part (a). On what input intervals is the graph of the function f increasing?

Part (b). On what input intervals is the graph of the function f decreasing?

Part (c). Are there any input values where the function f has a local maximum output? How did you decide?

Part (d). Are there any input values where the function f has a local maximum output? How did you decide?

Problem 7. The diagram below shows the graph of the *derivative* function for a function $y = f(x)$. Assume that f is continuous at the input value $x = 3$.



Graph of f'

Part (a). On what input intervals is the graph of the function f increasing?

Part (b). On what input intervals is the graph of the function f decreasing?

Part (c). Are there any input values where the function f has a local minimum output? How did you decide?

Part (d). Are there any input values where the function f has a local maximum output? How did you decide?

Problem 8. Let $y = f(x)$ be a function. Based on your work in Problems 6 and 7, which --- if any --- of the following statements is true?

- If the function f has a local maximum or local minimum output at an input value $x = a$, then it must be true that $f'(a) = 0$.
- If $f'(a) = 0$ for some input value $x = a$, then it must be true that the function f has a local maximum or local minimum output at $x = a$.

First Derivative Test

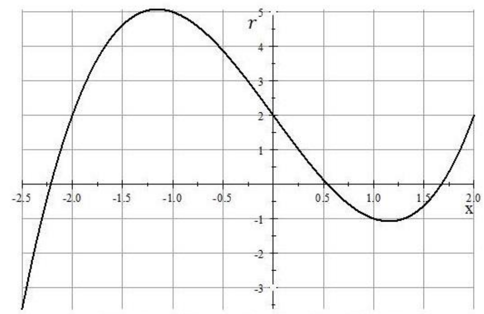
Let $y = f(x)$ be a function and let $r = f'(x)$ represent its derivative function. Suppose that $f(a)$ exists.

- If the output of f' changes from positive to negative at $x = a$, then the function f has a local maximum output at $x = a$.
- If the output of f' changes from negative to positive at $x = a$, then the function f has a local minimum output at $x = a$.

(This test assumes we are reading the graph from left to right.)

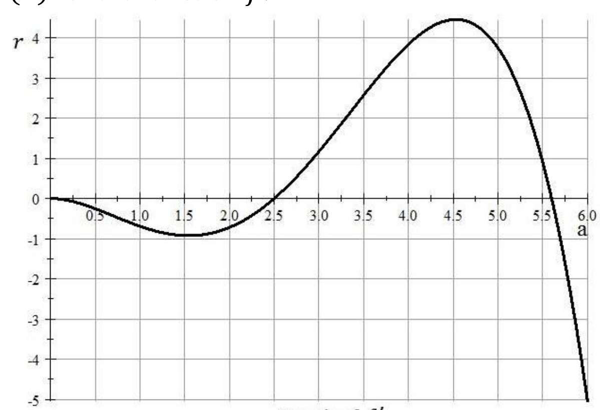
Homework.

Problem 1. The graph of the *derivative function* for a function $y = f(x)$ is shown below. Based on this graph, at what input values does the function f have local maximum or local minimum output?



Graph of the derivative function for f

Problem 2. The temperature T, measured in degrees Fahrenheit, for the hull of an airplane covaries with the altitude a of the plane, measured in thousands of feet above sea level. Suppose that $T = f(a)$ is the function that gives the values of T in terms of the values of a. The diagram below shows the graph of the *derivative* function $r = f'(a)$ for the function f.



Graph of f'

Part (a). The output variable for the function f' is called r. What are the units on this variable?

Part (b). What is the approximate value of $f'(3.0)$? What is the meaning of this number?

Part (c). If we assume the hull temperature of the plane is $55^{\circ}F$ when it is 3,000 feet above sea level, construct the formula for the tangent line to the graph of the function f at the point $(3.0, f(3.0))$.

Part (d). Based on the graph above, at what elevation above sea level will the airplane attain its maximum hull temperature?

Part (e). Based on the graph above, at what elevation above sea level will the airplane attain its minimum hull temperature?

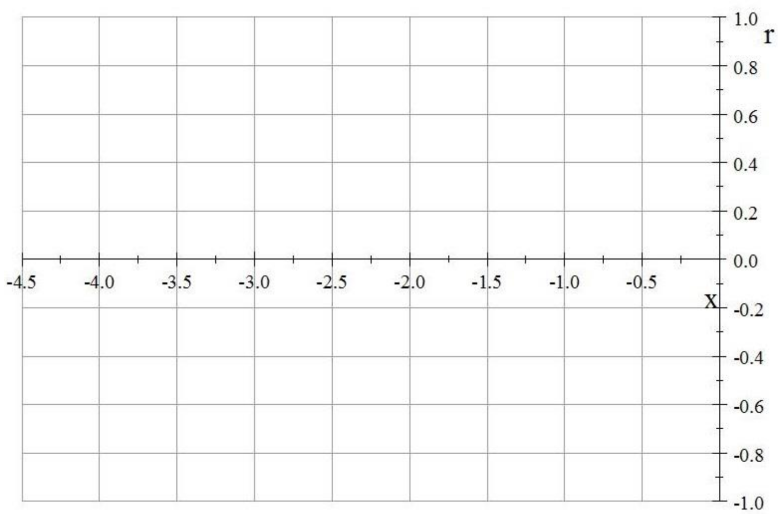
Problem 3. Consider the function $y = f(x)$ defined by the formula below.

$$f(x) = \frac{\ln(2 + \cos(x))}{2 + \sin(x)}$$

Part (a). Use the average rate of change function $r = \text{ARC}_{0.001}(x)$ for the function f to fill in the table below. (Be sure your calculator is in radian mode.)

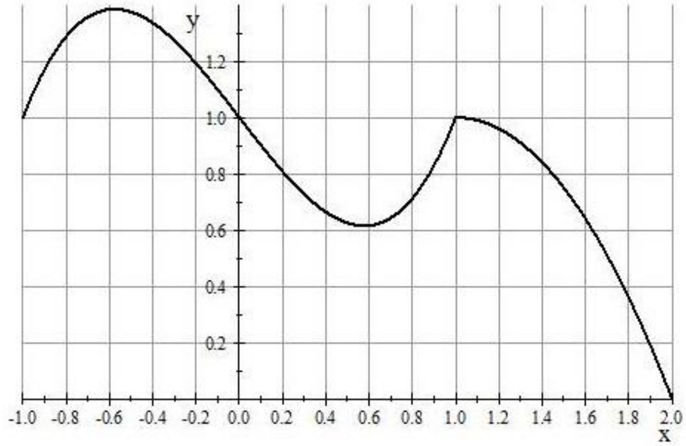
Input Value x	-4.50	-4.00	-3.50	-3.00	-2.50	-2.25	-2.00	-1.75	-1.50	-1.00	-0.50	0.00
Approximation for $f'(x)$												

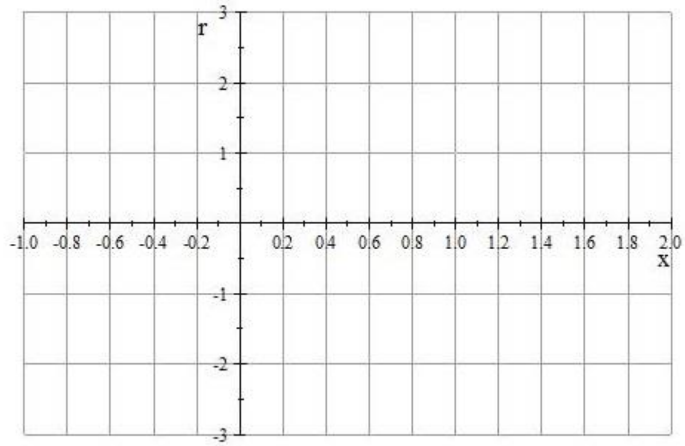
Part (b). Use your table values and the grid provided to sketch the graph of the derivative function $r = f'(x)$ for the function f.



Part (c). Use the information from your graph to construct the point-slope formula for the tangent line to the graph of the function f at the point $(-3.25, f(-3.25))$.

Problem 4. The graph of a function $y = f(x)$ is shown below. Use tangent lines to this graph to help sketch an approximation for the graph of $r = f'(x)$.





Use the general derivative rules and specific derivative formulas we have so far developed to differentiate the following functions.

(5) $f(x) = \dfrac{3}{e} - \dfrac{2}{x}$

(6) $g(u) = u\cos(u) - 3\sqrt{5}$

(7) $h(t) = e^t \sin(t) - t^3$

(8) $f(x) = \dfrac{\ln(x) - 2x}{5}$

(9) $g(u) = u^{-3}\ln(u)$

(10) $h(t) = t^7 \tan(t) - \sqrt{2}\, t$

(11) $f(x) = \dfrac{\pi}{x\sec(x)}$

(12) $g(u) = \dfrac{\sin(u) - \cos(u)}{u}$

(13) $h(t) = \dfrac{\cos(t)}{\sin(t)} - 3\sqrt{t}$

(14) $f(x) = \dfrac{2}{x} - x\tan(x)$

(15) $g(u) = u^2\sqrt{u}\, e^u$

(16) $h(t) = \dfrac{t^4 - t\cos(t)}{2t - 1}$

Use the anti-sum and anti constant-multiple rules along with the specific antiderivative formulas we have so far developed to determine the following arbitrary antiderivatives.

(17) $\int 2\pi\sqrt{7}\ dx$

(18) $\int 4\sec(u)\tan(u)\ du$

(19) $\int (3t - 4\cos(t))\ dt$

(20) $\int \left(\sec^2(x) - \frac{x^3}{\pi}\right) dx$

(21) $\int \left(\frac{8}{\sqrt{u}} - 2\ln(u)\right) du$

(22) $\int (e - 4\sin(t))\ dt$

(23) $\int \frac{\cos(x) - \sin(x)}{\sqrt{2}}\ dx$

(24) $\int (5\sec^2(u) + \ln(u))\ du$

(25) $\int \frac{t^3\sqrt{t} - 2}{\sqrt{t}}\ dt$

Answers to the Homework.

Problem 1. We see that the graph of the function f is increasing on the (approximate) input intervals $-2.6 < x < 0.55$ and $1.7 < x$. The graph of the function f is decreasing on the (approximate) input intervals $x < -2.6$ and $0.55 < x < 1.7$. The graph of f changes from increasing to decreasing at the input value $x \approx 0.55$; hence, the function f will have a local maximum output at this input value. By similar reasoning, the function f will have a local minimum output at the two input values $x \approx -2.6$ and $x \approx 1.7$.

Problem 2.

Part (a). The units are the same as those for the average rate of change --- Degrees Fahrenheit per 1000 feet of elevation above sea level.

Part (b). According to the graph, $f'(3.0) \approx 1.10$ degrees Fahrenheit per 1000 feet of elevation above sea level. This number tells us how small changes in the elevation quantity near 3,000 feet elevation will (approximately) affect changes in the temperature of the airplane. In particular, if h is close to 0, we know that

$$f(3.0 + h) \approx f(3.0) + 1.10 \cdot h$$

Part (c). The formula would be $T = 1.10(a - 3.0) + 55$.

Part (d). The First Derivative Test tells us the *local* maximum hull temperature will occur when the plan is about 5,600 feet above sea level. Considering the substantial *rate* of decrease in temperature for elevations greater than 5,600 feet, this local maximum is likely the greatest hull temperature the plane will experience. (We cannot be sure of this without more information.)

Part (e). The First Derivative Test tells us the *local* minimum hull temperature will occur when the plane is about 2,500 feet above sea level. However, this temperature is only a *local* minimum. Judging from the graph, it is likely that a lower hull temperature will occur at some elevation greater than 5,500 feet, since the *rate* of temperature decrease is substantial after this elevation.

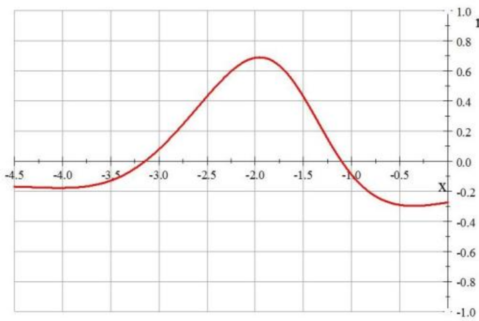
Problem 3.

Part (a). The data in the table below was obtained using the average rate of change function

$$\mathrm{ARC}_{0.001}(x) = \frac{f(x+0.001) - f(x)}{0.001}$$

Input Value x	-4.50	-4.00	-3.50	-3.00	-2.50	-2.25	-2.00	-1.75	-1.50	-1.00	-0.50	0.00
Approximation for $f'(x)$	-0.17	-0.178	-0.13	0.078	0.431	0.598	0.687	0.635	0.429	-0.09	-0.292	-0.275

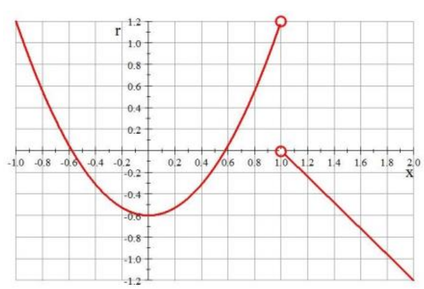
Part (b). Here is a sketch of the graph for the function $r = f'(x)$. Your sketch may vary slightly.



Part (c). First, note that $f(-3.25) \approx 0.00278$. Based on the graph above, we see that $f'(-3.25) \approx -0.10$. Therefore, the formula for the tangent line to the graph will be

$$y = T(x) \approx -0.10(x + 3.25) + 0.00278$$

Problem 4. Here is the approximate sketch of the derivative graph. Your graph may vary slightly.



(5) $\dfrac{df}{dx}(x) = \dfrac{d}{dx}\left[\dfrac{3}{e}\right] - 2\dfrac{d}{dx}\left[\dfrac{1}{x}\right]$ (Sum and Constant Multiple Rules)

$= 0 - 2\left(-\dfrac{1}{x^2}\right)$ (Specific Derivative Formulas)

$= \frac{2}{x^2}$

(6) $g'(u) = \frac{d}{du}[u\cos(u)] - \frac{d}{du}[3\sqrt{5}]$ (Sum Rule)

$= u\,\frac{d}{du}[\cos(u)] + \frac{d}{du}[u]\cos(u) - \frac{d}{du}[3\sqrt{5}]$ (Product Rule)

$= -u\sin(u) + \cos(u)$ (Specific Formulas)

(7) $\frac{dh}{dt}(t) = \frac{d}{dt}[e^t\sin(t)] - \frac{d}{dt}[t^3]$ (Sum Rule)

$= e^t\,\frac{d}{dt}[\sin(t)] + \frac{d}{dt}[e^t]\sin(t) - \frac{d}{dt}[t^3]$ (Product Rule)

$= e^t(\cos(t) + \sin(t)) - 2t^2$ (Specific Formulas)

(8) $f'(x) = \frac{1}{5}\left(\frac{d}{dx}[\ln(x)] - 2\frac{d}{dx}[x]\right)$ (Sum and Constant Multiple Rules)

$= \frac{1}{5x} - \frac{2}{5}$ (Specific Formulas)

(9) $\frac{dg}{du}(u) = u^{-3}\,\frac{d}{du}[\ln(u)] + \frac{d}{du}[u^{-3}]\ln(u)$ (Product Rule)

$= \frac{1 - 3\ln(u)}{u^4}$ (Specific Formulas and Simplification)

(10) $h'(t) = \frac{d}{dt}[t^7\tan(t)] - \sqrt{2}\frac{d}{dt}[t]$ (Sum and Constant Multiple Rules)

$= t^7\frac{d}{dt}[\tan(t)] + \frac{d}{dt}[t^7]\tan(t) - \sqrt{2}\frac{d}{dt}[t]$ (Product Rule)

$= t^7\sec^2(x) + 7t^6\tan(t) - \sqrt{2}$ (Specific Formulas)

(11) $f'(x) = \pi\frac{d}{dx}\left[\frac{1}{x\sec(x)}\right]$ (Constant Multiple Rule)

$= -\frac{\pi}{x^2\sin^2(x)}\,\frac{d}{dx}[x\sec(x)]$ (Reciprocal Rule)

$= -\frac{\pi}{x^2\sin^2(x)}\left(x\,\frac{d}{dx}[\sec(x)] + \frac{d}{dx}[x]\sec(x)\right)$ (Product Rule)

$= -\frac{\pi}{x^2\sin^2(x)}(x\sec(x)\tan(x) + \sec(x))$ (Specific Formulas)

(12) $\frac{dg}{du}(u) = \frac{1}{u^2}\left(u\frac{d}{du}[\sin(u) - \cos(u)] - (\sin(u) - \cos(u))\frac{d}{du}[u]\right)$ (Quotient Rule)

$= \frac{u\cos(u) + u\sin(u) - \sin(u) + \cos(u)}{u^2}$

(13) $h'(t) = \frac{d}{dt}\left[\frac{\cos(t)}{\sin(t)}\right] - 3\frac{d}{dt}[\sqrt{t}]$ (Sum and Constant Multiple Rules)

$= \frac{1}{\sin^2(t)}\left(\sin(t)\frac{d}{dt}[\cos(t)] - \cos(t)\frac{d}{dt}[\sin(t)]\right) - 3\frac{d}{dt}[\sqrt{t}]$ (Quotient Rule)

$= -\frac{1}{\sin^2(t)}(\sin^2(t) + \cos^2(t)) - \frac{3}{2\sqrt{t}}$ (Specific Formulas)

$= -\csc^2(t) - \frac{3}{2\sqrt{t}}$

(14) $\frac{df}{dx}(x) = 2\frac{d}{dx}\left[\frac{1}{x}\right] - \frac{d}{dx}[x\tan(x)]$ (Sum and Constant Multiple Rules)

$= 2\frac{d}{dx}\left[\frac{1}{x}\right] - \left(x\,\frac{d}{dx}[\tan(x)] + \frac{d}{dx}[x]\tan(x)\right)$ (Product Rule)

$= -\frac{2}{x^2} - x\sec^2(x) - \tan(x)$ (Specific Formulas)

(15) $g'(u) = u^2\,\frac{d}{du}[\sqrt{u}e^u] + \frac{d}{du}[u^2]\sqrt{u}e^u$ (Product Rule)

$= u^2\left(\sqrt{u}\,\frac{d}{du}[e^u] + \frac{d}{du}[\sqrt{u}]\,e^u\right) + \frac{d}{du}[u^2]\sqrt{u}e^u$ (Product Rule)

$= e^u\left(u^2\sqrt{u} + \frac{u^2}{2\sqrt{u}} + 2u\sqrt{u}\right)$ (Specific Formulas)

$= \frac{u^2\,e^u}{2\sqrt{u}}(2u+5)$

(16) $\frac{dh}{dt} = \frac{1}{(2t-1)^2}\left((2t-1)\frac{d}{dt}[t^4 - t\cos(t)] - (t^4 - t\cos(t))\frac{d}{dt}[2t-1]\right)$ (Quotient Rule)

$= \frac{1}{(2t-1)^2}\left((2t-1)\left[\frac{d}{dt}[t^4] - \frac{d}{dt}[t\cos(t)]\right] - (t^4 - t\cos(t))\left[2\frac{d}{dt}[t] - \frac{d}{dt}[1]\right]\right)$ (Sum and Constant Multiple Rules)

$= \frac{1}{(2t-1)^2}\left((2t-1)\left[\frac{d}{dt}[t^4] - \left(t\frac{d}{dt}[\cos(t)] + \frac{d}{dt}[t]\cos(t)\right)\right] - (t^4 - t\cos(t))\left[2\frac{d}{dt}[t] - \frac{d}{dt}[1]\right]\right)$ (Product Rule)

$= \frac{(2t-1)(4t^3 + t\sin(t) - \cos(t)) - 2(t^4 - t\cos(t))}{(2t-1)^2}$ (Specific Formulas)

(17) $\int 2\pi\sqrt{7}\;dx = 2\pi\sqrt{7}\,x + C$ (Specific antiderivative formula)

(18) $\int 4\sec(u)\tan(u)\;du = 4\int\sec(u)\tan(u)\;du$ (Anti CMR)

$= 4\sec(u) + C$ (Specific antiderivative formula)

(19) $\int (3t - 4\cos(t))\,dt = 3\int t\,dt - 4\int \cos(t)dt$ (Anti Sum and anti CMR)

$= \frac{3}{2}t^2 - 4\sin(t) + C$ (Specific antiderivative formulas)

(20) $\int \left(\sec^2(x) - \frac{x^3}{\pi}\right)\,dx = \int \sec^2(x)\;dx - \frac{1}{\pi}\int x^3\,dx$ (Anti Sum and anti CMR)

$= \tan(x) - \frac{x^4}{4\pi} + C$ (Specific antiderivative formulas)

(21) $\int \left(\frac{8}{\sqrt{u}} - 2\ln(u)\right)\,du = 8\int \frac{1}{\sqrt{u}}\,du - 2\int \ln(u)\,du$ (Anti Sum and anti CMR)

$= 16\sqrt{u} - 2(u\ln(u) - u) + C$ (Specific antiderivative formulas)

(22) $\int (e - 4\sin(t))\,dt = \int e\,dt - 4\int \sin(t)\,dt$ (Anti Sum and anti CMR)

$= e\,t + 4\cos(t) + C$ (Specific antiderivative formulas)

(23) $\int \frac{\cos(x) - \sin(x)}{\sqrt{2}}\,dx = \frac{1}{\sqrt{2}}\left(\int \cos(x)\,dx - \int \sin(x)\,dx\right)$ (Anti Sum and anti CMR)

$= \frac{\sin(x) + \cos(x)}{\sqrt{2}} + C$ (Specific antiderivative formulas)

(24) $\int (5\sec^2(u) + \ln(u))\;du = 5\int \sec^2(u)\;du + \int \ln(u)\,du$ (Anti Sum and anti CMR)

$= 5\tan(u) + u\ln(u) - u + C$ (Specific antiderivative formulas)

(25) $\int \frac{t^3\sqrt{t} - 2}{\sqrt{t}}\,dt = \int \left(t^3 - \frac{2}{\sqrt{t}}\right)\,dt$ (Rewriting fraction)

$= \int t^3\,dt - 2\int \frac{1}{\sqrt{t}}\,dt$ (Anti Sum and anti CMR)

$= \frac{t^4}{4} - 4\sqrt{t} + C$ (Specific antiderivative formulas)