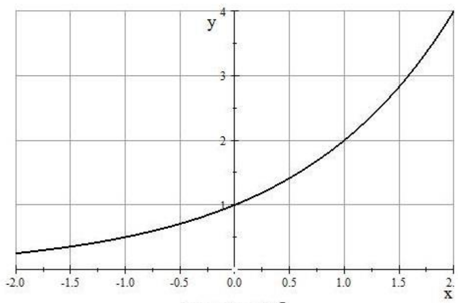
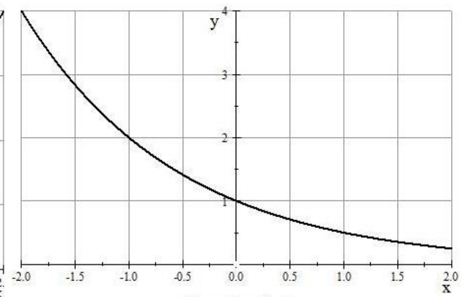
In the previous investigation, we noted that if $y = f(x)$ is a differentiable function, then the output sign of its derivative function gives important information about the graph of f. This derivative function has more to tell us, however, as the following investigation will show.

Problem 1. Consider the graph of the functions $y = f(x)$ and $y = g(x)$ shown below. Carefully draw several tangent lines to the graphs of each function.



Graph of f

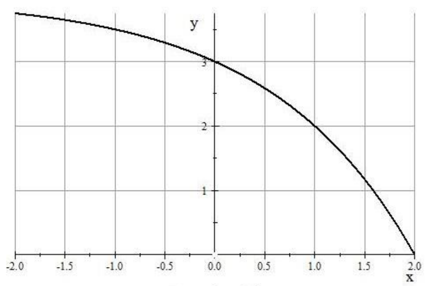


Graph of g

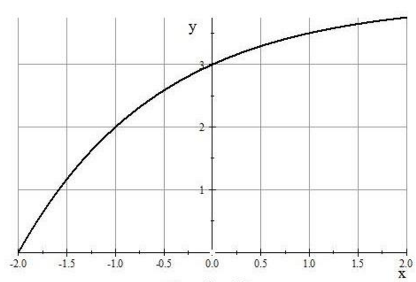
Part (a). What do you notice about the way the tangent lines you drew relate to the graphs?

Part (b). When you look at the tangent lines you drew in order from left to right, what do you notice about the constant rates of change for these lines?

Problem 2. Consider the graph of the functions $y = h(x)$ and $y = j(x)$ shown below. Carefully draw several tangent lines to the graphs of each function.



Graph of h



Graph of j

Part (a). What do you notice about the way the tangent lines you drew relate to the graphs?

Part (b). When you look at the tangent lines you drew in order from left to right, what do you notice about the constant rates of change for these lines?

Concavity

Let $y = f(x)$ be a differentiable function on an input interval $a < x < b$.

- We say that f is *concave up* on this interval provided the constant rates of change for the tangent lines to the graph of f are increasing from left to right.
- We say that f is *concave down* on this interval provided the constant rates of change for the tangent lines to the graph of f are decreasing from left to right.

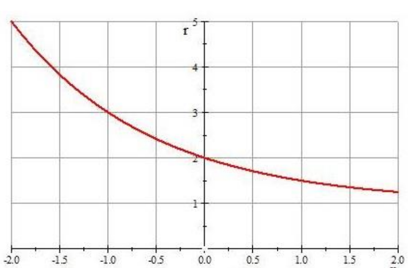
Problem 3. Suppose that $y = f(x)$ is a differentiable function on an input interval $a < x < b$. Complete the following sentences based on the definition of concavity.

- *The function f will be concave up on the input interval $a < x < b$ provided the derivative function f' is…*

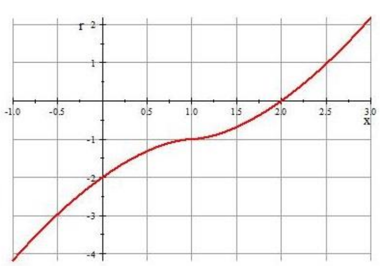
- *The function f will be concave down on the input interval $a < x < b$ provided the derivative function f' is…*

Problem 4. Let $m = f(t)$ represent the relationship between the mass m of carbon dioxide, measured in kilograms, emitted by the manufacturing plant Tox, Inc. and the number t of days since January 1, 2016. If the function f is differentiable and its graph is decreasing at an increasing rate, what can we way about the concavity of f? Explain your thinking.

Problem 5. Now, consider two different functions $y = f(x)$ and $y = g(x)$. The traces below show the graph of the *derivative* functions $r = f'(x)$ and $r = g'(x)$ for these functions. Looking at the derivative graphs, would you conclude the function f is concave up? What about the function g? How did you decide?



Graph of f'



Graph of g'

Second Derivative

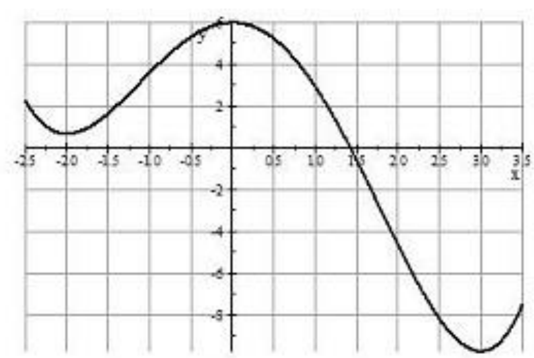
Let $y = f(x)$ be a function. The *second derivative* function for the function f is defined to be the derivative function for the derivative function $r = f'(x)$. The second derivative function provides the instantaneous rate of change for the derivative function at any input value. The second derivative function is commonly denoted by

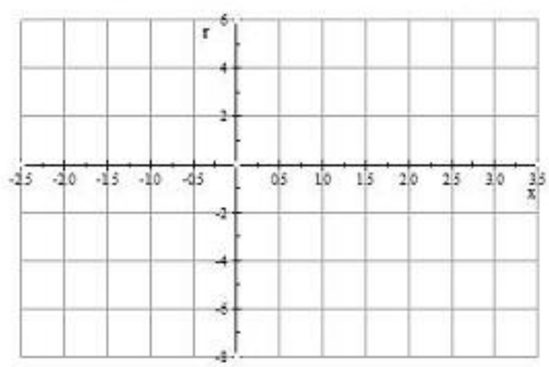
$$a = f''(x)$$

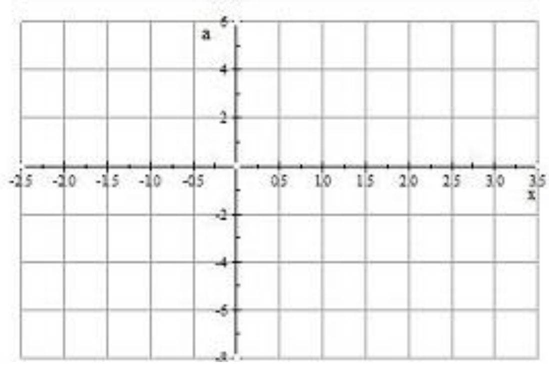
Problem 5. Suppose that $y = f(x)$ is a twice-differentiable function on an input interval $a < x < b$. Using the fact that $a = f''(x)$ is the derivative function for $r = f'(x)$, complete the following sentences.

- *The function f is concave up on the interval $a < x < b$ provided f'' …*
- *The function f is concave down on the interval $a < x < b$ provided f'' …*

Problem 6. The trace below shows the graph of a function $y = f(x)$. On the first grid, sketch the graph of the derivative function $r = f'(x)$ for the function f. Use your sketch of the derivative function f' to sketch the second derivative function $a = f''(x)$.

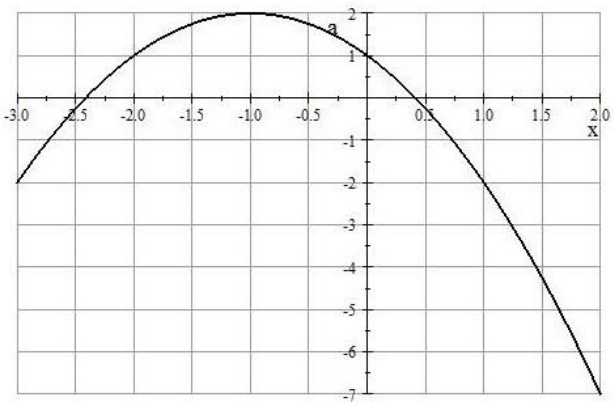






> ***Concavity and the Second Derivative***
>
> Suppose $y = f(x)$ is a twice-differentiable function on an input interval $a < x < b$.
>
> - If the output of the second derivative function is positive on an input interval, then the function is concave up on that input interval.
> - If the output of the second derivative function is negative on an input interval, then the function is concave down on that input interval.
> - If the output of the second derivative function changes sign at some input value $x = a$, then the function f has an *inflection point* at $x = a$. (That is, $x = a$ is an input value where the concavity of the function f changes.

Problem 7. The trace below shows the graph of the second derivative function for a function $y = f(x)$.



Graph of Second Derivative Function

Part (a). On which input intervals will the graph of the function f be concave up?

Part (b). On which input intervals will the graph of the function f be concave down?

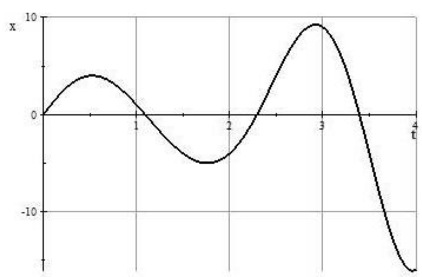
Part (c). At what input values will the function f have an inflection point?

Problem 8. Justo's cat, Mr. Hipsickles, is running back and forth chasing a fly. Let x represent the values of the quantity "Mr. Hipsickles' distance to the right of Justo, measured in feet" and let t represent the values of the quantity "the number of seconds passed since Mr. Hipsickles started chasing the fly." Let $x = f(t)$ represent the function that gives the values of x in terms of the values of t.

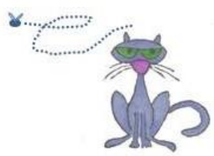
Part (a). If we let the first derivative function for f be represented by $r = f'(t)$, then what units will the output variable r have? What quantity values does r represent in this process?

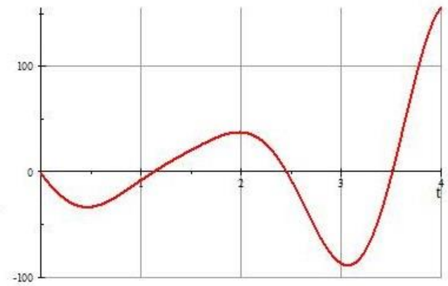
Part (b). If we let the second derivative function for f be represented by $a = f''(t)$, then what units will the output variable a have? What quantity values does a represent in this process?

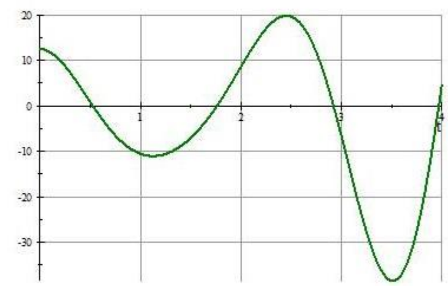
Part (c). The leftmost graph below shows the function $x = f(t)$. One of the other graphs shows the function $r = f'(t)$, and the other graph *may or may not* show the function $a = f''(t)$. Identify the graphs, and explain how you decided.



Graph of the Function f

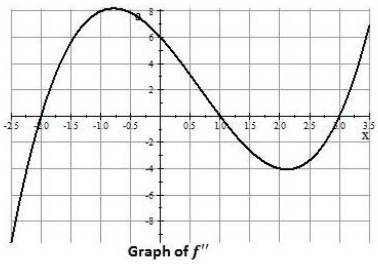




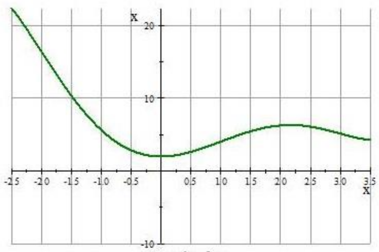


Homework.

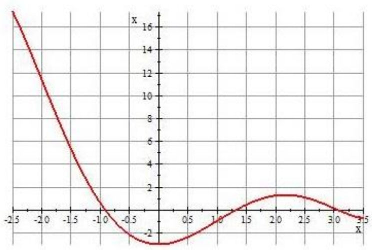
Problem 1. The leftmost graph below shows the second derivative function for some function $y = f(x)$. Could either of the graphs on the right represent the original function f? Justify your answer.



Graph of f''



Graph of g

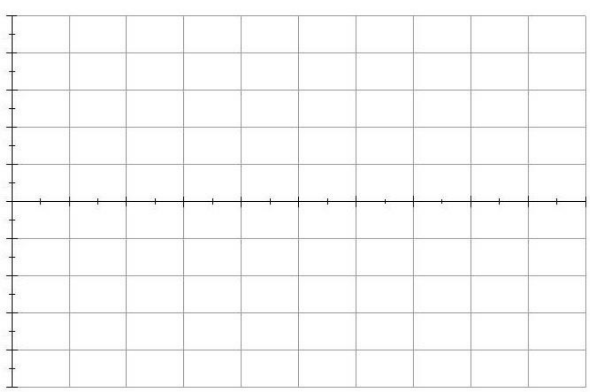


Graph of h

Problem 2. Tox, Inc. was recently cited for dumping polluted water into Lake Lotta Watta, causing a decline in the number of blue-finned mud twaddlers present in the lake. Tox, Inc. defended itself by stating that the population of mud twaddlers in the lake is “decreasing at a decreasing rate.”

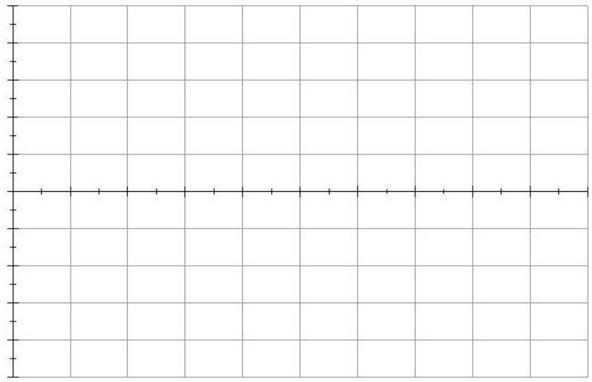
Part (a). Is this really a defense? Explain your thinking.

Part (b). Use the grid provided to sketch a graph that could give the population of blue-finned mud twaddlers as a function of time. Be sure to correctly identify the quantities used and label your axes accordingly.



Problem 3. On the grid provided, sketch the graph of a function $y = f(x)$ that satisfies the following six conditions. There are many possible answers. Be sure to label your axes.

- ... On the input interval $0 \leq x < 3$ we know $f'(x) < 0$.
- ... We know $f'(3) = 0$.
- ... On the input interval $3 < x \leq 4$ we know $f'(x) > 0$.
- ... On the input interval $0 \leq x < 1.5$ we know $f''(x) < 0$.
- ... We know $f''(1.5) = 0$.
- ... On the input interval $1.5 < x \leq 4$ we know $f''(x) > 0$.

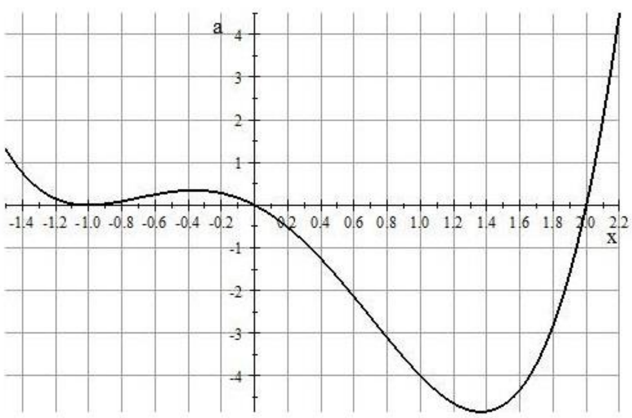


Problem 4. The trace below shows the graph of the second derivative function for a function $y = f(x)$.

Part (a). At which input values does the function f have inflection points?

Part (b). On which input intervals is the function f concave up?

Part (c). On which input intervals is the function f concave down?



Graph of f''

Compute the second derivative for each of the functions shown below.

(5) $f(x) = \cos(x)$ (6) $g(u) = u^3 + \ln(u)$ (7) $h(t) = 3\sin(t) - t^2$ (8) $j(y) = \dfrac{y^4}{e}$

(9) $f(x) = x\sin(x)$ (10) $g(u) = e^u \ln(u)$ (11) $h(t) = \dfrac{1+t}{t}$ (12) $j(y) = \sec(y)$

Problem 13. Consider the function $f(x) = x^4 + 4x^3 - 18x^2 + 8x - 5$.

Part (a). Determine the formula for f'' .

Part (b). Determine the input values where $f''(x) = 0$.

Part (c). Without graphing the function f, determine whether or not your solutions to Part (b) produce inflection points for the function f.

Problem 14. Consider the function $f(x) = 15x^7 - 42x^6 + \sqrt{5}$.

Part (a). Determine the formula for f'' .

Part (b). Determine the input values where $f''(x) = 0$.

Part (c). Without graphing the function f, determine whether or not your solutions to Part (b) produce inflection points for the function f.

Answers to the Homework.

Problem 1. The graph of the second derivative tells us the following about the function f.

- ... The graph of the function f must be concave down on the input intervals $x < -1$ and $1 < x < 3$.
- ... The graph of the function f must be concave up on the input intervals $-1 < x < 1$ and $3 < x$.
- ... The graph of the function f must have an inflection point at $x = -1$, $x = 1$, and $x = 3$.

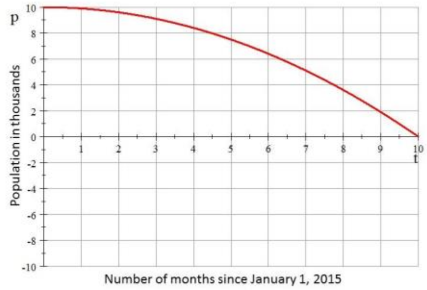
Inspection shows us that *both* graphs on the right satisfy these conditions (although the concavity of both graphs is debatable on the input interval $-2 < x < -1$). Therefore, the graph on the left *could* serve as the second derivative for both of the functions whose graphs are shown on the right.

Problem 2.

Part (a). To say that the population is decreasing at a decreasing rate implies that the population graph is decreasing *and* concave down. This is not really a defense, unless you want the mud twaddlers to go extinct.

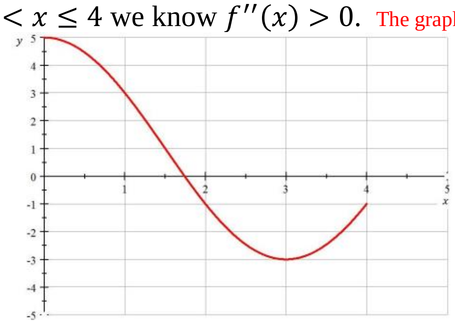
Part (b). Here is one rather pessimistic example.

Let t represent the number of months passed since January 1, 2015, and let p represent the population of mud twaddlers present in the lake, measured in thousands.



Problem 3. The graph shown below is one of many that satisfy the criteria laid out in this problem.

- ..... On the input interval $0 \leq x < 3$ we know $f'(x) < 0$. The graph of f is decreasing on this input interval.
- ... We know $f'(3) = 0$.
- ... On the input interval $3 < x \leq 4$ we know $f'(x) > 0$. The graph of f is increasing on this input interval.
- ... On the input interval $0 \leq x < 1.5$ we know $f''(x) < 0$. The graph of f is concave down on this input interval.
- ... We know $f''(1.5) = 0$.
- ... On the input interval $1.5 < x \leq 4$ we know $f''(x) > 0$. The graph of f is concave up on this input interval.



Problem 4.

Part (a). The function f will have inflection points at $x = 0$ and at $x = 2$. The function will *not* have an inflection point at $x = -1$ because the output sign of the second derivative function does not change.

Part (b). The function f will be concave up on the input intervals $x < -1$, $-1 < x < 0$, and $2 < x$.

Part (c). The function will be concave down on the input interval $0 < x < 2$.

(5) $f'(x) = -\sin(x)$ $\qquad$ $f''(x) = \frac{d}{dx}[-\sin(x)] = -\frac{d}{dx}[\sin(x)] = -\cos(x)$

(6) $g'(u) = \frac{d}{du}[u^3] + \frac{d}{du}[\ln(u)] = 3u^2 + \frac{1}{u}$

$$g''(u) = 3\frac{d}{du}[u^2] + \frac{d}{du}\left[\frac{1}{u}\right] = 6u - \frac{1}{u^2}$$

(7) $h'(t) = 3\dfrac{d}{dt}[\sin(t)] - \dfrac{d}{dt}[t^2] = 3\cos(t) - 2t$

$h''(t) = 3\dfrac{d}{dt}[\cos(t)] - 2\dfrac{d}{dt}[t] = -3\sin(t) - 2$

(8) $j'(y) = \dfrac{1}{e} \cdot \dfrac{d}{dy}[y^4] = \dfrac{4y^3}{e}$ $\qquad j''(y) = \dfrac{4}{e} \cdot \dfrac{d}{dy}[y^3] = \dfrac{12y^2}{e}$

(9) $f'(x) = x \cdot \dfrac{d}{dx}[\sin(x)] + \dfrac{d}{dx}[x] \cdot \sin(x) = x\cos(x) + \sin(x)$

$f''(x) = x \cdot \dfrac{d}{dx}[\cos(x)] + \dfrac{d}{dx}[x] \cdot \cos(x) + \dfrac{d}{dx}[\sin(x)] = -x\sin(x) + 2\cos(x)$

(10) $g'(u) = e^u \cdot \dfrac{d}{du}[\ln(u)] + \dfrac{d}{du}[e^u] \cdot \ln(u) = e^u\left(\dfrac{1}{u} + \ln(u)\right)$

$g''(u) = e^u \cdot \left(\dfrac{d}{du}\left[\dfrac{1}{u}\right] + \dfrac{d}{du}[\ln(u)]\right) + \dfrac{d}{du}[e^u] \cdot \left(\dfrac{1}{u} + \ln(u)\right) = e^u\left(\ln(u) + \dfrac{2}{u} - \dfrac{1}{u^2}\right)$

(11) $h'(t) = \dfrac{1}{t^2}\left(t \cdot \dfrac{d}{dt}[1+t] - (1+t) \cdot \dfrac{d}{dt}[t]\right) = -\dfrac{1}{t^2}$

$h''(t) = -\dfrac{d}{dt}[t^{-2}] = \dfrac{2}{t^3}$

(12) $j'(y) = \sec(y)\tan(y)$

$j''(y) = \sec(y) \cdot \dfrac{d}{dy}[\tan(y)] + \dfrac{d}{dy}[\sec(y)] \cdot \tan(y) = \sec(y)\,(\sec^2(y) + \tan^2(y))$

Problem 13.
Part (a). $f''(x) = 12x^2 + 24x - 36 = 12(x^2 + 2x - 3)$

Part (b). $0 = f''(x) \quad \Rightarrow \quad 0 = x^2 + 2x - 3 \quad \Rightarrow \quad 0 = (x-1)(x+3) \quad \Rightarrow \quad x = 1$ and $x = -3$

Part (c). For these input values to produce inflection points for the function f, we know that the output sign of f'' must change from positive to negative (or vice-versa) at these inputs. Since we have identified the input values where $f''(x) = 0$, and since the function f'' has no input values where it's output is undefined, we know the output of f'' must have the same sign (either positive or negative) in each input interval bounded by $x = -3$ and $x = 1$. We can therefore determine whether the output sign of f'' changes at these inputs by simply picking a convenient point in each interval bounded by these values and testing them.

- In the interval $-\infty < x < -3$, consider the "test value" $x = -4$. Note that $f''(-4) > 0$.
- In the interval $-3 < x < 1$, consider the "test value" $x = 0$. Note that $f''(0) < 0$.
- In the interval $1 < x < +\infty$, consider the "test value" $x = 2$. Note that $f''(2) > 0$.

Based on this information, we can see that the output sign of f'' changes at $x = -3$ and $x = 1$. Hence, we may conclude that the function f has an inflection point at both.

Problem 14.
Part (a). $f''(x) = 630x^5 - 1260x^4 = 630x^4(x-2)$

Part (b). $0 = f''(x) \quad \Longrightarrow \quad 0 = x^4(x-2) \;\; \Longrightarrow \;\; x = 0$ and $x = 2$

Part (c). The input values from Part (b) determine three input intervals.

- In the interval $-\infty < x < 0$, consider the "test value" $x = -1$. Note that $f''(-1) < 00$.
- In the interval $0 < x < 2$, consider the "test value" $x = 1$. Note that $f''(1) < 0$.
- In the interval $2 < x < +\infty$, consider the "test value" $x = 3$. Note that $f''(3) > 0$.

Based on this information, we can see that the output sign of f'' changes at $x = 2$. Therefore, we know that f has a single inflection point, namely at the input value $x = 2$.