In this investigation, we introduce one of the most powerful tools in calculus.

Problem 1. Use the Product Rule to differentiate the function.

$$y = f(x) = \cos^2(x) = \cos(x) \cdot \cos(x)$$

Now, let's think differently about the function f appearing in Problem 1. This function is the *composition* of two functions, namely

$$u = g(x) = \cos(x) \quad \text{and} \quad m(u) = u^2$$

In other words, $f(x) = (\cos(x))^{\,2} = m\big(g(x)\big)$.

Problem 2. Consider the functions g and m defined above.

Part (a). Differentiate each function with respect to its input variable.

$\dfrac{dg}{dx}(x) =$

$\dfrac{dm}{du}(u) =$

Part (b). Construct the formula for the function $j(x) = m'\big(g(x)\big) \cdot g'(x)$. How does this formula compare to the derivative function $f'(x)$ you computed in Problem 1?

The Chain Rule for Derivatives

If it is possible to write a function $y = f(x)$ as the composition of two differentiable functions $y = m(u)$ and $u = g(x)$, then f is a differentiable function of the input variable x, and

$$\frac{df}{dx}(x) = m\big(g(x)\big) \cdot g'(x) = \left.\frac{dm}{du}\right|_{u=g(x)} \cdot \frac{dg}{dx}(x)$$

The vertical line in the formula means "Input $g(x)$into the function $\frac{dm}{du}$." It is commonly used because the usual function notation $\frac{dm}{du}\big(g(x)\big)$ is visually confusing.

Example 1. Use the Chain Rule to differentiate $f(x) = \sec^3(x)$ with respect to the variable x.

Solution. First, observe that $f(x) = [\sec(x)]^3$; consequently, f is constructed by composing two differentiable functions, namely

$$u = g(x) = \sec(x) \qquad \text{and} \qquad y = m(u) = u^3$$

Since $f(x) = m(g(x))$, the Chain Rule therefore tells us

$$\frac{df}{dx}(x) = \left.\frac{dm}{du}\right|_{u=\mathrm{g}(x)} \cdot \frac{dg}{dx}(x)$$

The specific derivative formulas tell us

$$\left.\frac{dm}{du}\right|_{u=g(x)} = \left.\frac{d}{du}[u^3]\right|_{u=\sec(x)} = 3(\sec(x))^2 \qquad \text{and} \qquad \frac{dg}{dx}(x) = \frac{d}{dx}[\sec(x)] = \sec(x)\tan(x)$$

Thus, we know

$$\frac{df}{dx}(x) = \left.\frac{dm}{du}\right|_{u=\sec(x)} \cdot \frac{dg}{dx}(x) = 3(\sec(x))^2 \cdot (\sec(x)\tan(x)) = 3\sec^3(x)\tan(x)$$

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Problem 3. Consider the function $f(x) = \tan(x^{-4})$.

Part (a). Identify a functions $u = g(x)$ and $y = m(u)$ such that $f(x) = m\big(g(x)\big)$.

Part (b). Construct the following formulas.

$$\frac{dg}{dx}(x) = \qquad\qquad \left.\frac{dm}{du}\right|_{u=g(x)} =$$

$$\frac{df}{dx}(x) =$$

Problem 4. Use the Chain Rule to differentiate the function $f(x) = [x^2 + 3\sin(x)]^4$.

$$\frac{dg}{dx}(x) = \qquad\qquad \left.\frac{dm}{du}\right|_{u=g(x)} =$$

$$\frac{df}{dx}(x) =$$

Problem 5. Use the Chain Rule to differentiate the function $f(x) = \cos(\sin(x))$.

Problem 6. For any fixed positive number a, the Base-a exponential function $y = f(x) = a^x$ can be written as

$$y = f(x) = e^{x\ln(a)}$$

Use the Chain Rule to compute $\dfrac{d}{dx}[a^x]$.

Technical details of its proof notwithstanding, the beauty of the Chain Rule lies in the fact that we may use it to differentiate a function with respect to *any variable we choose*. We simply assume that the function is a composite function of the variable we have chosen.

Example 2. Differentiate the function $f(x) = \tan(x)$ with respect to the variable t.

Solution. We treat the function f as a composite function of the variable t. In other words, we assume $x = g(t)$, where g is some unknown (but differentiable) function of t. Differentiation then proceeds as always with the Chain Rule.

$$\frac{df}{dt} = \frac{d}{dx}[\tan(x)] \cdot \frac{dg}{dt} = \sec^2(x)\frac{dx}{dt}$$

Since we don't have the explicit formula for the function g, we are done.[1]

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The process we used in Example 2 is often referred to as *implicit differentiation*, since we are assuming that the input variable is some implicit function of the differentiation variable.

Problem 7. Differentiate the function $f(x) = x\sin(x)$ with respect to the variable t.

[1] When working with the Chain Rule, it is common to write $\frac{dx}{dt}$ rather than the more precise $\frac{dx}{dt}(t)$.

Problem 8. Consider the function $f(u) = \sqrt{u}$.

Part (a). Differentiate this function with respect to the variable x.

Part (b). Suppose we also know that $u = g(x) = x + \cos(x)$. Write $\frac{df}{dx}$ as an explicit function of x.

Part (c). Now, suppose we instead know that $u = g(x) = 3 - 2^x$. Write $\frac{df}{dx}$ as an explicit function of x.

Homework.

Problem 1. Differentiate the function $f(x) = \frac{2x+\sqrt{x}}{\pi}$ with respect to the variable t.

Problem 2. Differentiate the function $f(u) = u\tan(u)$ with respect to the variable x.

Each of the following functions is composite with respect to the input variable x. Use the Chain Rule to differentiate each of these functions with respect to x.

(3) $f(x) = \sqrt{\tan(x)}$

(4) $f(x) = \cos^{-4}(x)$

(5) $f(x) = \sec(3x + x^3)$

(6) $f(x) = 4^{2+\sin(x)}$

(7) $f(x) = 7(e^x + x)^{3/2}$

(8) $f(x) = \sin(\sin(x))$

Problem 9. Differentiate the function $f(x) = x^2\tan(x^3)$ with respect to the variable x.

Problem 10. Differentiate the function $f(x) = \frac{\cos(x^2)}{x}$ with respect to the variable x.

Problem 11. Let $f(x) = (x^2 - 6x + 8)^{1/2}$.

Part (a). Use the Chain Rule to construct the derivative function $r = f'(x)$.

Part (b). For what values of the input variable x will $f'(x) = 0$?

We know that $\frac{d}{dx}[\ln(x)] = x^{-1}$. Does this mean $F(x) = \ln(x)$ is an antiderivative for $f(x) = x^{-1}$? The answer is "NOT QUITE" because the function f is defined for all *nonzero* input values x, but the function F is only defined for *positive* input values x and therefore could serve as an antiderivative for f only when $x > 0$. There is a way around this problem, however.

Problem 12. Consider the piecewise-defined function

$$G(x) = \begin{cases} \ln(x) & \text{if} \quad x > 0 \\ \ln(-x) & \text{if} \quad x < 0 \end{cases}$$

Part (a). Show that $G'(x) = f(x)$ for all nonzero values of x. (Hence, G is one antiderivative for f.)

Part (b). Explain why we may now conclude $\int \frac{1}{x} dx = \ln|x| + C$.

Problem 13. Samuel places an ice cube on the kitchen table, and it starts to melt. The weight $W = H(t)$ of the ice cube, measured in ounces, *changes* with respect to the time t, measured in minutes, since the ice cube was placed on the table according to the formula

$$\frac{dH}{dt}(t) = H'(t) = 1 - \frac{10}{e^t}$$

Of course, the function H is *one* antiderivative for the function H'.

Part (a). Consider the function $F(t) = t + 10e^{-t}$. By differentiating this function, show that F is *also* an antiderivative for the function H'.

Part (b). After three minutes, Samuel weighs the ice cube and finds that it weighs 7.75 ounces. Explain why this tells us that $F(t) \neq H(t)$.

Part (c). Since both functions F and H are antiderivatives for H', there must exist a constant C such that $H(t) = F(t) + C$. What is the approximate value of this constant?

Part (d). Suppose that Samuel misread the scale when he weighed the ice cube. Six minutes after placing the ice cube on the table, he re-measures the weight and determines that the ice cube weighs 5.25 ounces. Based on this new information, what is the value of the constant C needed to make $H(t) = F(t) + C$?

Answers to the Homework.

Problem 1. First, observe that

$$\frac{df}{dx} = \frac{1}{\pi}\left(2\frac{d}{dx}[x] + \frac{d}{dx}[\sqrt{x}]\right) = \frac{1}{\pi}\left(2 + \frac{1}{2\sqrt{x}}\right) = \frac{4\sqrt{x}+1}{2\sqrt{x}\pi}$$

Treating x as some implicit function of the variable t, we have

$$\frac{df}{dt} = \frac{df}{dx}\cdot\frac{dx}{dt} = \left(\frac{4\sqrt{x}+1}{2\sqrt{x}\pi}\right)\frac{dx}{dt}$$

Problem 2. First, observe that

$$\frac{df}{du} = \frac{d}{du}[u]\cdot\tan(u) + u\frac{d}{du}[\tan(u)] = \tan(u) + u\sec^2(u)$$

Treating u as some implicit function of the variable x, we have

$$\frac{df}{dx} = \left(\tan(u) + u\sec^2(u)\right)\frac{du}{dx}$$

Problem 3. In this case, we see that f can be explicitly written as a composite function of the differentiation variable. In particular, if we let $u = g(x) = \tan(x)$ and let $y = h(u) = \sqrt{u}$, then we know

$$\frac{df}{dx} = \frac{d}{du}[\sqrt{u}]\Big|_{u=\tan(x)}\cdot\frac{d}{dx}[\tan(x)] = \frac{1}{2\sqrt{\tan(x)}}\cdot\sec^2(x) = \frac{\sec^2(x)}{2\sqrt{\tan(x)}}$$

Problem 4. In this case, we see that f can be explicitly written as a composite function of the differentiation variable. In particular, if we let $u = g(x) = \cos(x)$ and let $y = h(u) = u^{-4}$, then we know

$$\frac{df}{dx} = \frac{d}{du}[u^{-4}]\Big|_{u=\cos(x)}\cdot\frac{d}{dx}[\cos(x)] = -4(\cos(x))^{-5}\cdot(-\sin(x)) = \frac{4\sin(x)}{\cos^5(x)}$$

Problem 5. In this case, we see that f can be explicitly written as a composite function of the differentiation variable. In particular, if we let $u = g(x) = 3x + x^3$ and let $y = h(u) = \sec(u)$, then we know

$$\frac{df}{dx} = \frac{d}{du}[\sec(u)]\Big|_{u=3x+x^3}\cdot\frac{d}{dx}[3x + x^3] = \sec(3x + x^3)\tan(3x + x^3)\cdot(3 + 3x^2)$$

Problem 6. In this case, we see that f can be explicitly written as a composite function of the differentiation variable. In particular, if we let $u = g(x) = 2 + \sin(x)$ and let $y = h(u) = 4^u$, then we know

$$\frac{df}{dx} = \frac{d}{du}[4^u]\Big|_{u=2+\sin(x)}\cdot\frac{d}{dx}[2 + \sin(x)] = 4^{2+\sin(x)}\ln(4)\cdot\cos(x)$$

Problem 7. In Homework Problem 10 of Investigation 12, you showed that

$$\frac{d}{du}\left[u^{3/2}\right] = \frac{3}{2}\sqrt{u}$$

If we let $u = g(x) = e^x + x$ and let $y = h(u) = 7u^{3/2}$, then we know

$$\frac{df}{dx} = 7\frac{d}{du}\left[u^{3/2}\right]\Big|_{u=e^x+x} \cdot \frac{d}{dx}[e^x + x] = \frac{21}{2}\sqrt{e^x + x} \cdot (e^x + 1)$$

Problem 8. In this case, we see that f can be explicitly written as a composite function of the differentiation variable. In particular, if we let $u = g(x) = \sin(x)$ and let $y = h(u) = \sin(u)$, then we know

$$\frac{df}{dx} = \frac{d}{du}[\sin(u)]\Big|_{u=\sin(x)} \cdot \frac{d}{dx}[\sin(x)] = \cos(\sin(x)) \cdot \cos(x)$$

Problem 9. In this problem we will need to apply the Product Rule, followed by the Chain Rule.

$$\frac{df}{dx} = \frac{d}{dx}[x^2]\tan(x^3) + x^2\frac{d}{dx}[\tan(x^3)]$$

$$= \frac{d}{dx}[x^2] \cdot \tan(x^3) + x^2\frac{d}{du}[\tan(u)]\Big|_{u=x^3} \cdot \frac{du}{dx} \qquad \text{(Let } u = x^3\text{)}$$

$$= 2x\tan(x^3) + 3x^4\sec^2(x^3)$$

Problem 10. In this problem, we will need to apply the Quotient Rule, followed by the Chain Rule.

$$\frac{df}{dx} = \left(\frac{1}{x^2}\right)\left(x\frac{d}{dx}[\cos(x^2)] - \frac{d}{dx}[x]\cos(x^2)\right)$$

$$= \left(\frac{1}{x^2}\right)\left(x \cdot \frac{d}{du}[\cos(u)]\Big|_{u=x^2} \cdot \frac{du}{dx} - \frac{d}{dx}[x] \cdot \cos(x^2)\right) \qquad \text{(Let } u = x^2\text{)}$$

$$= -\frac{2x^2\sin(x^2) + \cos(x^2)}{x^2}$$

Problem 11.

Part (a). Observe

$$\frac{df}{dx} = \frac{d}{du}\left[u^{1/2}\right]\Big|_{u=x^2-6x+8} \cdot \frac{d}{dx}[x^2 - 6x + 8] \qquad \text{Let } u = x^2 - 6x + 8$$

$$= \frac{x-3}{\sqrt{x^2 - 6x + 8}}$$

Part (b). We will have $f'(x) = 0$ only when $x - 3 = 0$; that is, only when $x = 3$.

Problem 12.

Part (a). If $x > 0$, then $G(x) = \ln(x)$, and there is nothing to show. On the other hand, if $x < 0$, then we know $G(x) = \ln(-x)$. Observe that

$$\frac{d}{dx}[\ln(-x)] = \frac{d}{du}[\ln(u)]\Big|_{u=-x} \cdot \frac{d}{dx}[-x] \qquad \text{Let } u = -x$$

$$= \frac{1}{-x} \cdot (-1)$$

$$= \frac{1}{x}$$

Part (c). First, since we have shown that G is one antiderivative for the function f, we may write

$$\int x^{-1} dx = G(x) + C$$

If $x > 0$, then $\ln|x| = \ln(x)$; and, if $x < 0$, then $\ln|x| = \ln(-x)$. Consequently, we know $G(x) = \ln|x|$.

Problem 13.

Part (a). Observe

$$\frac{d}{dt}[t + 10e^{-t}] = \frac{d}{dt}[t] + 10\frac{d}{dt}[e^{-t}]$$

$$= \frac{d}{dt}[t] + 10 \cdot \frac{d}{du}[e^{u}]\Big|_{u=-t} \cdot \frac{d}{dt}[-t] \qquad \text{Let } u = -t$$

$$= 1 + 10 \cdot e^{-t} \cdot (-1)$$

$$= 1 - 10e^{-t}$$

Part (b). We know that $F(3) = 3 + 10e^{-3} \approx 3.4978$ ounces. Since by measuring, Samuel knows $H(3) = 7.75$ ounces, these two functions cannot be the same.

Part (c). We know that $7.75 = H(3) = F(3) + C$. Hence, $C = 7.75 - F(3) \approx 4.2522$ ounces.

Part (d). We know that $5.25 = H(3) = F(3) + C$. Hence, $C = 5.25 - F(3) \approx 1.7522$ ounces.