In the previous investigation, we explored the Chain Rule for differentiating composite functions, but we offered little motivation for why we should differentiate composite functions using this process. In this investigation, we will look more closely at how the average rate of change functions for a composite function can be related to the average rate of change functions for its components.

Let $y = f(u)$ and let $u = g(x)$ be functions. Recall that the *composite* function $f \circ g$ is defined by the rule

$$[f \circ g](x) = f\big(g(x)\big)$$

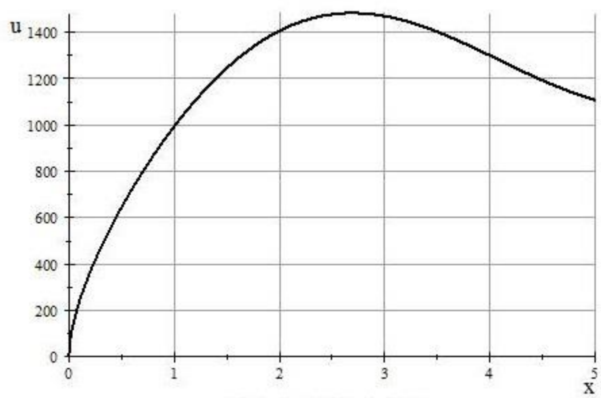
Consequently, we know that average rate of change functions for $f \circ g$ are given by the general formula

$$r = ARC_h(x) = \frac{f\big(g(x+h)\big) - f\big(g(x)\big)}{h}$$

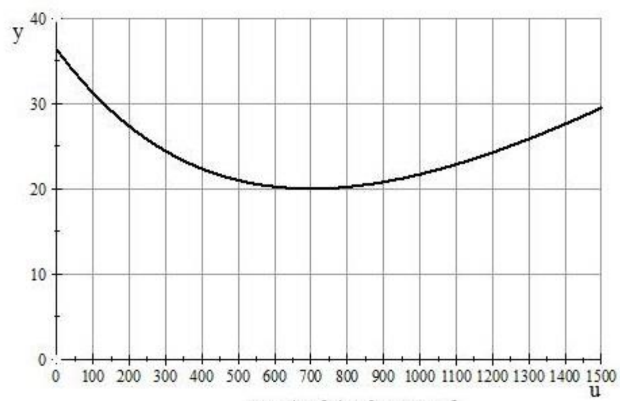
However, there is quite a bit more that we can say. Let's consider an example.

The temperature, measured in degrees Fahrenheit, for the hull of a jetliner depends on the altitude of the plane. Let $y = f(u)$ represent the function that gives the temperature of the jetliner in degrees Fahrenheit in terms of the altitude above sea level, measured in feet, of the jetliner.

Furthermore, the altitude of the jetliner above sea level depends on the time since the jetliner left the airport. Let $u = g(x)$ represent the function that gives the altitude of the jetliner above sea level in terms of the time since it left the airport, measured in minutes.



Graph of the function g

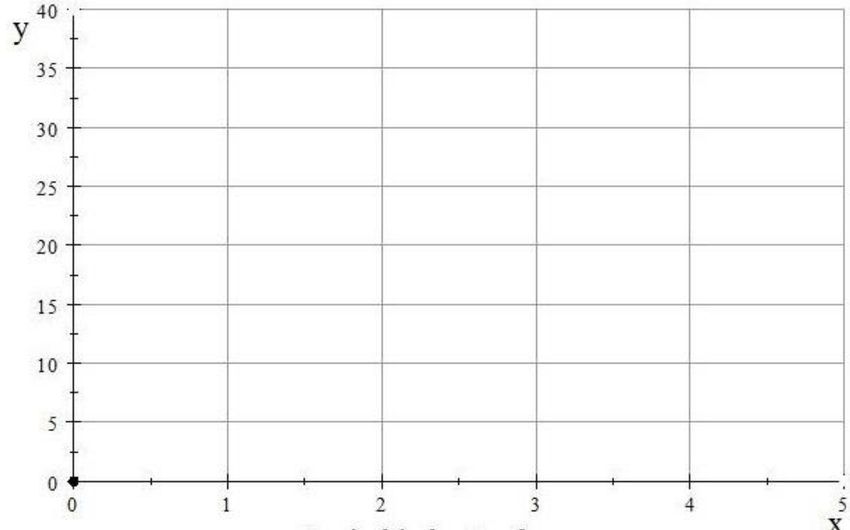


Graph of the function f

Problem 1. Use the graphs of the functions g and f above to fill in the table.

Value of x	0.0	0.5	1.0	1.5	2.0	3.0	4.0	5.0
Value of $g(x)$								
Value of $f\big(g(x)\big)$								

Problem 2. Use the table of values from Problem 1 to help you sketch the graph of the composite function $f \circ g$ on the grid provided.



Graph of the function $f \circ g$

Problem 3. Consider your graph of the function $f \circ g$ from Problem 2. What is the approximate average rate of change this function on the input interval from $x = 0.5$ minutes to $x = 2.0$ minutes?

$\text{ARC}_{1.5}^{(f \circ g)}(0.5) \approx$

Problem 4. Now, consider the graph of the function g.What is the approximate average rate of change for the function g on the input interval from $x = 0.5$ minutes to $x = 2.0$ minutes?

$$\text{ARC}_{1.5}^{(g)}(0.5) = \frac{\Delta_y\big(g(0.5), g(2.0)\big)}{\Delta_x(0.5,2.0)} \approx$$

Problem 5. Now, consider the graph of the function f. What is the average rate of change for the function f on the input interval from from $u = 600$ feet above sea level to $u = 1400$ feet above sea level?

$$\text{ARC}_{800}^{(f)}(600) = \frac{\Delta_y\big(f(600), f(1400)\big)}{\Delta_u(600,1400)} \approx$$

Problem 6. Is the following equation true? Does it agree with your computations in Problems 3, 4, and 5? Explain your thinking.

$$\mathrm{ARC}_{1.5}^{(f\circ g)}(0.5) \approx \frac{\Delta_y\Big(f\big(g(0.5)\big), f\big(g(2.0)\big)\Big)}{\Delta_u\big(g(0.5), g(2.0)\big)} \cdot \frac{\Delta_u\big(g(0.5), g(2.0)\big)}{\Delta_x(0.5{,}2.0)}$$

Problem 7. Now, let h be any small nonzero real number, and let $j_h = \Delta_u\big(g(0.5), g(0.5+h)\big)$. If we let h get closer and closer to 0, will it also be true that j_h gets closer and closer to 0? Explain your thinking.

Note that $g(0.5+h) = g(0.5) + j_h$. Based on our work in this investigation, it now seems reasonable to write

$$\begin{aligned}
[f\circ g]'(0.5) &= \lim_{h\to 0} \mathrm{ARC}_h^{(f\circ g)}(0.5) \\
&= \lim_{h\to 0}\left[\frac{\Delta_y\Big(f\big(g(0.5)\big), f\big(g(0.5+h)\big)\Big)}{\Delta_u\big(g(0.5), g(0.5+h)\big)} \cdot \frac{\Delta_u\big(g(0.5), g(0.5+h)\big)}{\Delta_x(0.5{,}0.5+h)}\right] \\
&= \lim_{h\to 0}\left[\frac{f\big(g(0.5+h)\big) - f\big(g(0.5)\big)}{j_h} \cdot \frac{g(0.5+h) - g(0.5)}{h}\right] \\
&= \lim_{h\to 0}\left[\frac{f\big(g(0.5)+j_h\big) - f\big(g(0.5)\big)}{j_h} \cdot \frac{g(0.5+h) - g(0.5)}{h}\right] \\
&= f'\big(g(0.5)\big)\cdot g'(0.5))
\end{aligned}$$