In calculus, we are interested in how changes in the values of variables affect the values themselves. We should note that the word "change" has lots of meanings in English. For example, when we use "change" in a sentence, we could mean

- Replacing something with something else: "I changed the sheets on the bed today."
- An ongoing process: "This caterpillar is changing into a butterfly."
- The result of a completed process: "The value of x changed from 1 to 2."

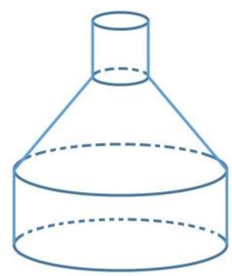
Consequently, we will limit use of the word "change" and focus on the word "vary" when we are discussing a process.[1]

Important Terms Associated with a Varying Quantity

Suppose that u represents the values of measures for a varying quantity in a process.

1. When we say that u *varies in an interval*, we mean that u takes on at least one value from this interval during the process.
2. The *variation* of u in an interval is the *process* of u varying in that interval.

Wilhelmina purchased the beaker shown below at a flea market. The beaker is twelve inches tall and is capable of holding one gallon of liquid. Wilhelmina places the beaker under a faucet and begins to fill it at a constant rate of 0.10 gallons per second.



Problem 1. Imagine the process of water filling the beaker.

Part (a). Do you think there is a time measure when the volume of water in the flask measures exactly $\pi/4$ gallons? Explain your thinking.

Part (b). Do you think there is a volume measure when the depth of water measures exactly $\sqrt[3]{10}$ inches? Explain your thinking.

Part (c). Let t represent measures of the quantity "time passed since the faucet was turned on, measured in seconds." Is the volume of water in the beaker increasing at the rate of 0.10 gallons per second in the time interval $1.025 \leq t \leq 1.026$ seconds? Explain your thinking.

[1] The content of this investigation is motivated by the opening activities in Project DIRACC (P. Thompson & M. Ashbrook) http://patthompson.net/ThompsonCalc/index.html.

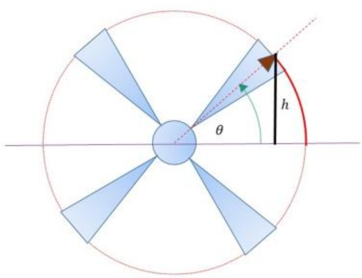
Part (d). Let V represent measures of the quantity "the volume of water in the beaker, measured in gallons." At what rate is water flowing into the beaker for each of the following intervals?

$$0 \le V \le 1 \qquad 0.45 \le V \le 0.73 \qquad 0.001 \le V \le 0.0011$$

Function

Suppose x and y represent the values of varying quantities in a process, and suppose x varies in an interval I and y varies in an interval J. We say that y is a *function* of x in the interval I provided there is some action R which, as x varies in the interval I, assigns each value of x in I to *exactly one* value of y in J. The rule R is called a *function action* (or simply a *function*) from I to J.

The propeller of a single-engine plane is slowly rotating counterclockwise as the pilot prepares for takeoff. Each blade is one meter long, and there is a small chip on the tip of one blade. This particular blade was at the 3:00 position when the propeller began to rotate. Let θ represent measures of the quantity "the radian measure of the angle between the chip's initial and current position," and let h represent measures of the quantity "the vertical distance to the chip above the horizontal, measured in meters.



Imagine θ varying in the interval $[0, 3\pi/2]$. The variable h is a function of the variable θ in this interval; let's let R represent the function action.

Problem 2. What is the largest value of h that R can assign to a value of θ?

Problem 3. What is the smallest value of h that R can assign to a value of θ?

Problem 4. It is customary to let the symbol $R(\theta)$ represent the value of h that is assigned by the function action R to the value of θ. This symbol means "The *output* value of the function action R for the *input* value θ" --- or more simply "R of θ."

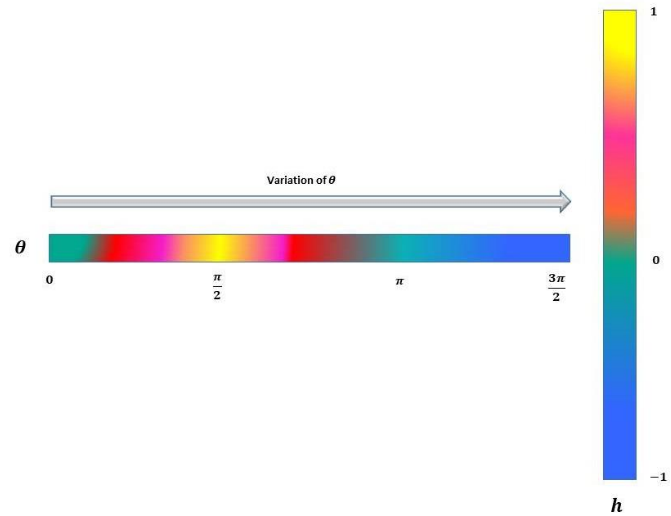
Part (a). In the interval $[0, 3\pi/2]$, what value (or values) of θ give us $R(\theta) = 0$?

Part (b). What is the output value $R(\pi/2)$? What is the output value $R(3\pi/2)$?

Problem 5. The function action R is a dynamic relationship between the variable θ and the variable h. As θ varies in the interval $[0,3\pi/2]$, the function action forces h to vary in the interval $[-1,1]$, coordinating each value of θ with a value of h. The diagram below is color-coded to show how the function action R as it coordinates the values of θ and h.

In this diagram, as θ varies in the interval $[0,3\pi/2]$, the colors appearing indicate which value of h is assigned to each value of θ. For example, since $R(0) = 0$ and $R(\pi) = 0$, both $\theta = 0$ and $\theta = \pi$ have been given the color corresponding to $h = 0$.

Place your left finger at $\theta = 0$ and your right finger at $h = 0$ in the diagram. Moving your left finger across the interval $[0,3\pi/2]$ at a constant speed, trace out the variation of h with your right finger by coordinating colors.

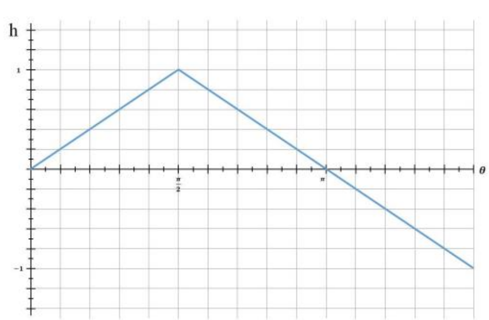


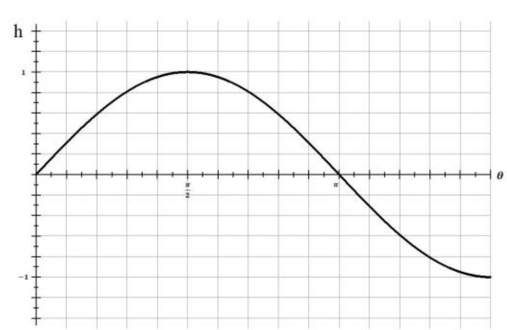
Function Graph

Suppose x and y represent the values of varying quantities in a process, and suppose y is a function of x in the interval I. If we let R represent the function action, then the *graph* of R is a visual trace that coordinates the values of x with the values of y as x varies in the interval I.

To create the graph of a function action R, it is customary to interpret the input interval I as a horizontal number line and imagine x varying from left to right in I. To capture the function action, we interpret the output interval J as a vertical number line that intersects the horizontal number line at $x = 0$. As x varies in I, we place the ordered pair $(x, R(x))$ on the vertical line passing through the value x.

Problem 6. Which of the diagrams below do you think best represents the graph of the function action R? Carefully explain your thinking.

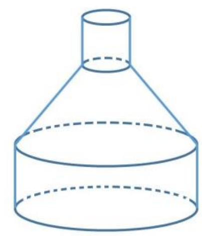




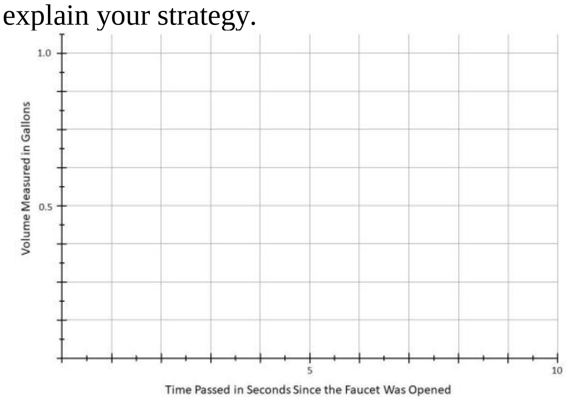
Problem 7. Do you think that θ is a function of h in the interval $[-1,1]$? Carefully explain your thinking.

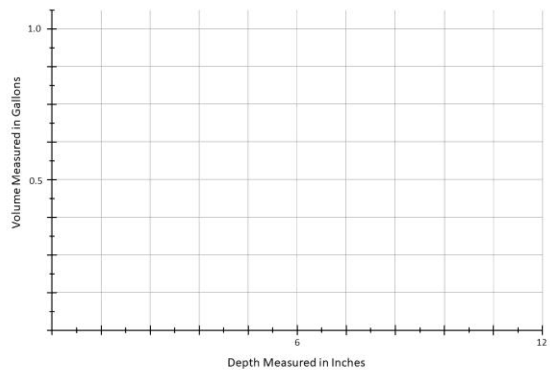
Problem 8. Let's think again about Wilhelmina and her beaker from Problem 1. Assume water is entering the beaker at a constant rate of 0.10 gallons per minute.

- Let Let t represent measures of the quantity "time passed since the faucet was turned on, measured in seconds."
- Let V represent measures of the quantity "the volume of water in the beaker, measured in cubic inches."
- Let h represent measures of the quantity "the depth of water in the beaker, measured in inches."

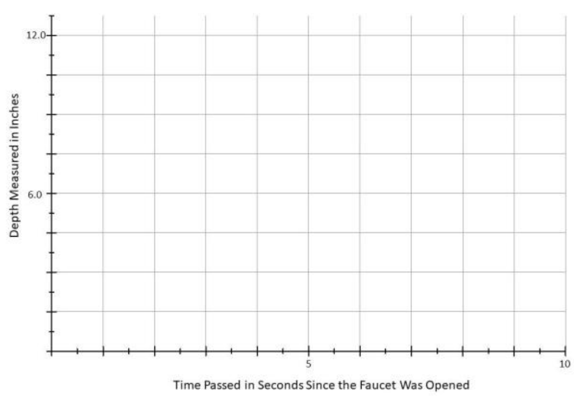


Part (a). Using the grids below, sketch traces that you think would reasonably describe the function $V = f(t)$ in the interval $0 \le t \le 10$ and the function $V = g(h)$ in the interval $0 \le h \le 12$. Be prepared to explain your strategy.





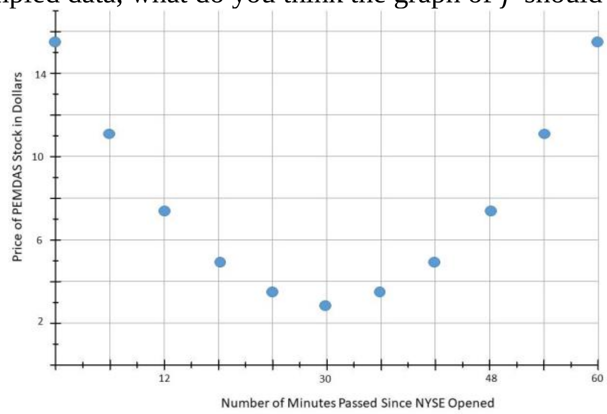
Part (b). Use the grid below to sketch a trace that could reasonably describe the function $h = m(t)$ in the interval $0 \le t \le 10$. Be prepared to explain your strategy.



PEMDAS, an international shipping company, has publicly traded stock listed on the New York Stock Exchange (NYSE). Its stock is traded online, and computer-generated trades happen very quickly. Let t represent the time passed in minutes since NYSE opened on Tuesday, and let p represent the price of PEMDAS stock, measured in dollars.

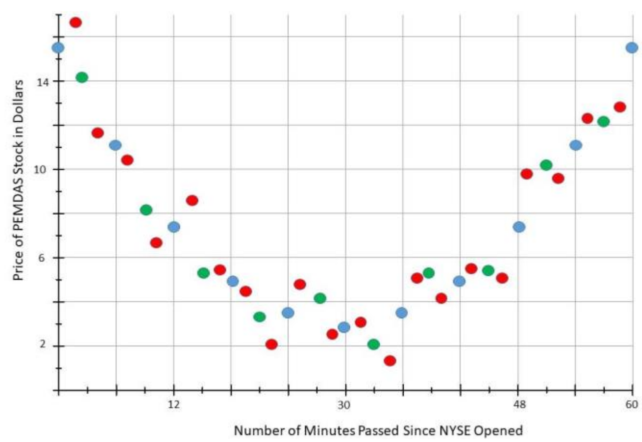
Problem 9. The diagram below gives points on the graph of the function $p = f(t)$ for the first hour that NYSE was open. In this case, the output of the function is recorded at the beginning of six-minute time intervals.

Part (a). Based on the sampled data, what do you think the graph of f should reasonably be?



Part (b). Based on your graph, what to what do you predict would be the value of PEMDAS stock 4π minutes after NYSE opened? Explain your thinking.

Problem 10. The diagram below shows points on the graph of the function $p = f(t)$ for the first hour that NYSE was open. In this case, the output of the function is recorded at the beginning of ninety-second time intervals.



Part (a). Based on this information, was the graph you drew in Problem 9 an accurate depiction of the function $p = f(t)$? Explain.

Part (b). Based on this information, would you change your prediction about the value of $f(4\pi)$? Explain your thinking.

Computer-generated stock trades can happen *thousands* of times per minute, so there is a good chance that even the data points shown above do not accurately capture the covariation between p and t in the interval $0 \leq t \leq 60$ minutes.

Fundamental Rule of Covariation
Suppose a represents measures of a varying quantity in a process, and suppose $a = f(b)$ in some interval. In calculus, whenever you consider a varying in that input interval, *always* assume a can also vary in any subinterval of that interval --- no matter how small. Furthermore, assume variations in the smaller input intervals provide important information about the output.

Homework.

Problem 1. Suppose that m and n represent measures of varying quantities in a process, and suppose that n is a function of m in the interval $-3 \leq m \leq 2$. Let the function action be defined by the formula $n = R(m) = 4 - \sqrt{m^2 + 1}$ in the interval $-3 \leq m \leq 2$.

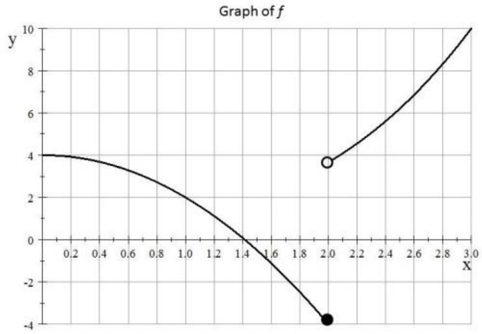
Part (a). Does R represent a variable in this process? Explain.

Part (b). Does $R(m)$ represent a variable in this process? Explain.

Part (c). Are there any input values m in the interval $[-3,2]$ such that $R(m) = 0$? If so, what are these values?

Part (d). Are there any input values m in the interval $[-3,2]$ such that $R(m) = 1$? If so, what are these values?

Problem 2. Suppose that x and y represent measures of varying quantities in a process, and suppose $y = f(x)$ in the interval $0 \le x \le 3$. Assume that the function rule is defined by the graph shown below.



Part (a). What is the output value assigned to the input value $x = 2.0$?

Part (b). If we let a be any value in the interval $0 \le x \le 1.0$, is it necessarily true that $f(a + 1.5) = f(a) + f(1.5)$?

Part (c). Is it true that $\dfrac{f(1.0)}{f(2.0)} = \dfrac{1.0}{2.0}$?

Part (d). Is it true that $f\big(f(0.8)\big) = [f(0.8)]^2$?

Warning About Function Notation

If f represents a function rule, then f is *not* a variable, and you cannot assume it will follow the same rules that variables follow in algebraic expressions. In particular, unless you have explicit evidence to the contrary, you must assume

$$f(a+b) \neq f(a) + f(b) \qquad f(ab) \neq f(a)f(b) \qquad \frac{f(a)}{f(b)} \neq \frac{a}{b} \qquad f\big(f(a)\big) \neq [f(a)]^2$$

Problem 3. Saluda bought a \$22,000 car from Dewey Cheatham and financed the car for five years (sixty months). Her first car payment was due on January 12.

- Let m represent the values of the quantity "The number of months passed since January 12."
- Let c represent the amount of money, measured in dollars, that Salida owes on her car."

Part (a). Is the variable c a function of the variable t in the interval $[0,60]$?

Part (b). Is the variable t a function of the variable c in the interval $[0,22{,}000]$?

Problem 4. Suppose that Saluda only makes car payments on the twelfth day of each month. The table below gives the amount owed on her car on this date for January through April of her first year.

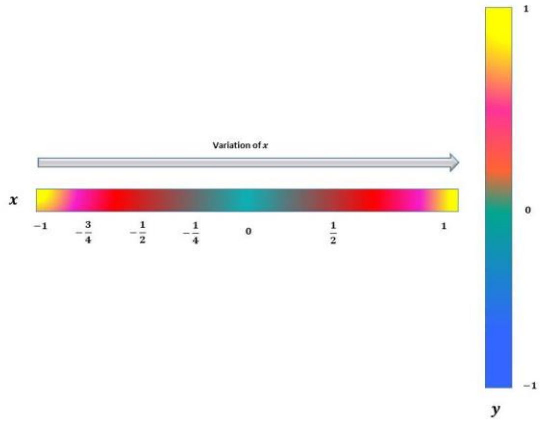
Value of t	0	1	2	3
Value of c	21,817	21,635	21,454	21,274

Part (a). Does c vary in the interval $[21500,22000]$? Explain.

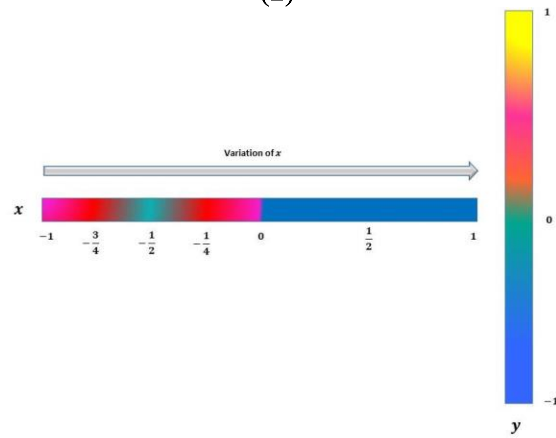
Part (b). Does c vary in the interval $[21650,21800]$? Explain.

Problem 5. Each of the color-coded variations shown below represents a function $y = f(x)$ from the interval $-1 \le x \le 1$ to the interval $-1 \le y \le 1$. Match each of these variations to the trace which best describes the graph of the function. Not all traces will be used.

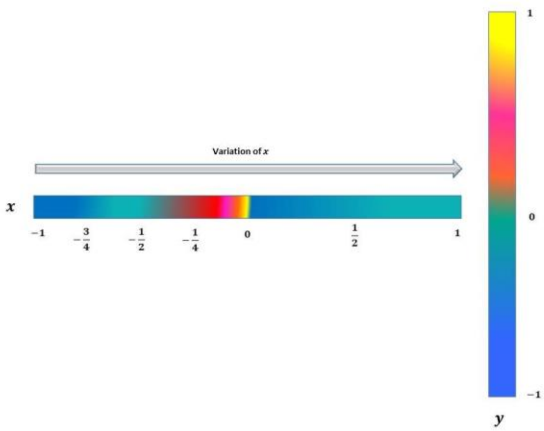
(1)



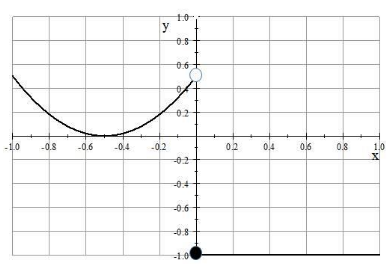
(2)



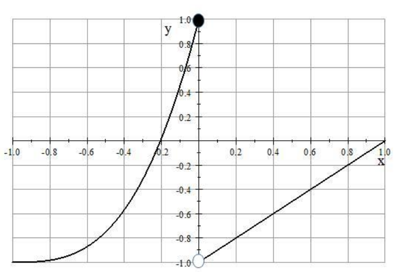
(3)



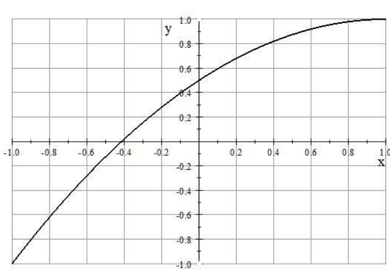
(A)



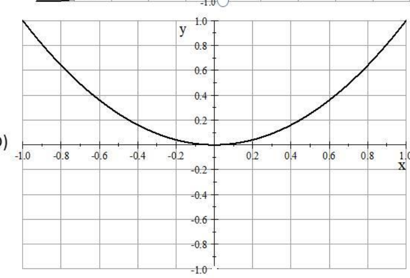
(B)



(C)



(D)



Problem 6. Boris places his goldfish, Natasha, in a cylindrical glass bowl having a radius of three inches, and containing twelve inches of water. Natasha is one-half inch long and one-eighth of an inch wide. Boris proceeds to watch Natasha swim around for five minutes.

- Let t represent the values of the quantity "The number of minutes passed since Boris started watching Natasha."
- Let x represent the values of the quantity "The horizontal distance between Natasha's center and the central axis of the bowl, measured in millimeters."
- Let h represent the values of the quantity "The vertical distance between Natasha's center and the bottom of the bowl, measured in millimeters."

Part (a). Does t vary in the interval $[3.75, 3.76]$?

Part (b). Does x vary in the interval $[1.4, 1.45]$?

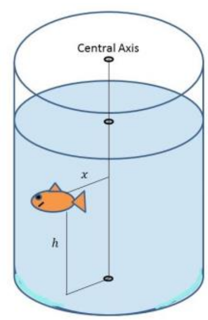
Part (c). Does x vary in the interval $[2.94, 2.95]$?

Problem 7. The diagram below shows Natasha in her bowl.

Part (a). Is the variable h a function of the variable t in the interval $[0,5]$?

Part (b). Is the variable t a function of the variable h in the interval $[0,12]$?

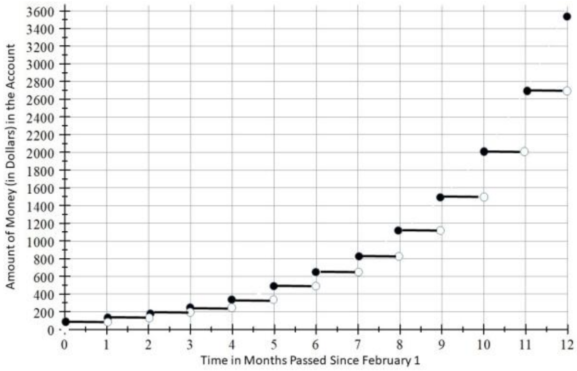
Part (c). Is the variable h a function of the variable x in the interval $[0,3]$?

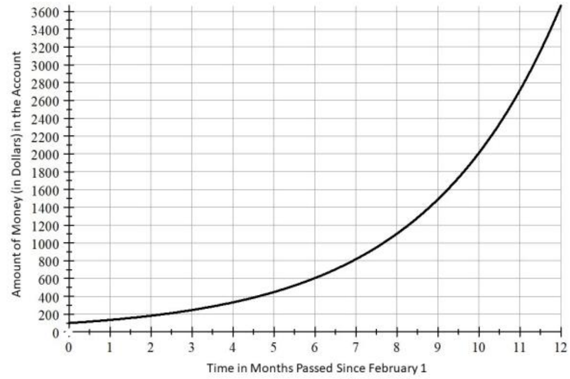


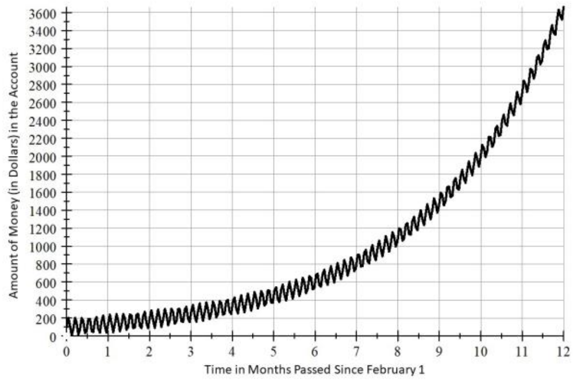
Problem 8. Suppose Terrance deposits \$100 into an account at the Incredible Bank of Murfreesboro (IBM) on February 1. The table below shows the amount of money in his account, rounded to the nearest dollar, at the end of each month passed since February 1.

Month	1	2	3	4	5	6	7	8	9	10	11	12
Amount in the Account	135	182	246	332	448	605	817	1,103	1,489	2,011	2,714	3,664

Let A represent measures of the quantity "the amount of money in the account, measured in dollars"; and let t represent measures of the quantity "the time passed since February 1, measured in months." The following traces represent potential graphs of the function $A = f(t)$ in the interval $0 \le t \le 12$. In each case, construct a reasonable scenario that would make the trace correct.







Problem 9. Suppose that p and w represent the measures of two varying quantities in a process, and suppose that

$$w = h(p) = 2p^2 - 1$$

Part (a). What does the letter h represent in this relationship?

Part (b). What is the input variable for this relationship?

Part (c). What is the output variable for this relationship?

Part (d). Write down the formula for the expression

$$\frac{h(p+1)}{h(p)+1}$$

Problem 10. Suppose that r and s represent the measures of two varying quantities in a process, and suppose that

$$r = j(s) = \sqrt{2s+4}$$

Part (a). What does the letter j represent in this relationship?

Part (b). What is the input variable for this relationship?

Part (c). What is the output variable for this relationship?

Part (d). What is the value of the expressions $j(6) \cdot j(6)$ and $j\big(j(6)\big)$?

Answers to the Homework.

Homework.

Problem 1. Suppose that m and n represent measures of varying quantities in a process, and suppose that n is a function of m in the interval $-3 \le m \le 2$. Let the function action be defined by the formula $n = R(m) = 4 - \sqrt{m^2 + 1}$ in the interval $-3 \le m \le 2$.

Part (a). Does R represent a variable in this process? Explain.

No. The letter "R" represents the function action.

Part (b). Does $R(m)$ represent a variable in this process? Explain.

Yes. The expression "$R(m)$" represents the output value of the function action R for the input value m; in particular, $R(m)$ is a value of the variable n.

Part (c). Are there any input values m in the interval $[-3,2]$ such that $R(m) = 0$? If so, what are these values?

First, observe that

$$0 = R(m) \quad \Rightarrow \quad 0 = 4 - \sqrt{m^2 + 1} \quad \Rightarrow \quad 16 = m^2 + 1 \quad \Rightarrow \quad \pm\sqrt{15} = m$$

However, neither of these solutions lies in the specified input interval; hence the answer is "no".

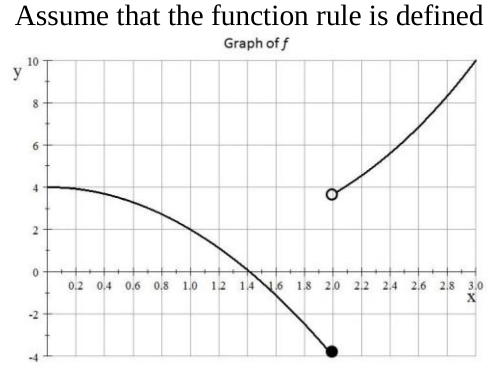
Part (d). Are there any input values m in the interval $[-3,2]$ such that $R(m) = 1$? If so, what are these values?

First, observe that

$$1 = R(m) \quad \Rightarrow \quad 1 = 4 - \sqrt{m^2 + 1} \quad \Rightarrow \quad 9 = m^2 + 1 \quad \Rightarrow \quad \pm 2\sqrt{2} = m$$

Only the solution $m = -2\sqrt{2}$ lies in the specified input interval.

Problem 2. Suppose that x and y represent measures of varying quantities in a process, and suppose $y = f(x)$ in the interval $0 \le x \le 3$. Assume that the function rule is defined by the graph shown below.



Part (a). What is the output value assigned to the input value $x = 2.0$? $f(2.0) = -4$

Part (b). If we let a be any value in the interval $0 \le x \le 1.0$, is it necessarily true that $f(a + 1.5) = f(a) + f(1.5)$?

No. Consider $a = 0$. In this case, $f(a + 1.5) = f(1.5) \approx 1.5$, but $f(0) + f(1.5) \approx 4 + 1.5 = 5.5$. The equation will fail for many other values of a as well.

Part (c). Is it true that $\dfrac{f(1.0)}{f(2.0)} = \dfrac{1.0}{2.0}$? $\dfrac{f(1.0)}{f(2.0)} = \dfrac{2}{-4} = -\dfrac{1}{2} \neq \dfrac{1}{2}$

Part (d). Is it true that $f\big(f(0.8)\big) = [f(0.8)]^2$?

No. We have $f\big(f(0.8)\big) = f(1.0) = 2$ while $[f(0.8)]^2 = [1.0]^2 = 1$.

Warning About Function Notation

If f represents a function rule, then f is *not* a variable, and you cannot assume it will follow the same rules that variables follow in algebraic expressions. In particular, unless you have explicit evidence to the contrary, you must assume

$$f(a+b) \neq f(a) + f(b) \qquad f(ab) \neq f(a)f(b) \qquad \frac{f(a)}{f(b)} \neq \frac{a}{b} \qquad f\big(f(a)\big) \neq [f(a)]^2$$

Problem 3. Saluda bought a $22,000 car from Dewey Cheatham and financed the car for five years (sixty months). Her first car payment was due on January 12.

- Let m represent the values of the quantity "The number of months passed since January 12."
- Let c represent the amount of money, measured in dollars, that Salida owes on her car."

Part (a). Is the variable c a function of the variable t in the interval $[0,60]$?

It is reasonable to assume that there can be only one amount of money Saluda could owe on her car in any given month since January 12. Therefore, the variable c is a function of the variable t.

Part (b). Is the variable t a function of the variable c in the interval $[0,22,000]$?

The question as to whether the variable t is a function of the variable c is harder to answer without more information. However, it *seems* reasonable to assume that the amount Saluda will owe on her car will never increase or stay the same as the months pass. If we make this assumption, then there only be one (at most) one month in which Saluda could owe any given amount of money on her car; and t would be a function of c. If this assumption proves incorrect, then this would not be the case.

Problem 4. Suppose that Saluda only makes car payments on the twelfth day of each month. The table below gives the amount owed on her car on this date for January through April of her first year.

Value of t	0	1	2	3
Value of c	21,817	21,635	21,454	21,274

Part (a). Does c vary in the interval $[21500,22000]$? Explain.

Yes; the variable c takes on four values in this interval.

Part (b). Does c vary in the interval $[21650,21800]$? Explain.

No; the variable c does not take on any values in this interval.

Problem 5. Each of the color-coded variations shown below represents a function $y = f(x)$ from the interval $-1 \leq x \leq 1$ to the interval $-1 \leq y \leq 1$. Match each of these variations to the trace which best describes the graph of the function. Not all traces will be used.

Let's consider the covariation represented by (1). Based on the color coding of the y-values, we can see that, as x varies from $x = -1$ to $x = 1$, the output of f decreases from $f(-1) = 1$ to $f(0) = 0$ and then increases again to $f(1) = 1$. Therefore, the trace which best fits is (D).

Let's consider the covariation represented by (2). Based on the color coding of the y-values, we can see that, as x varies from $x = -1$ to $x = 0$, the output of f first decreases from $f(-1) \approx 0.5$ to a minimum of $f(-1/2) = 0$ and then increases again to $f(0) \approx 0.5$. There is a sudden change in output at the input value $x = 0$; indeed, it is possible that $f(0) = -1$. (There is not enough information to know for sure; however, we can be certain there is a "jump" in the trace at the input value $x = 0$.) The output of the function then remains constant on the input interval $0 < x \leq 1$. Therefore, the trace which best fits is (A).

Let's consider the covariation represented by (3). Based on the color coding of the y-values, we can see that, as x varies from $x = -1$ to $x = 0$, the output of f increases from $f(-1) = -1$ to a maximum output of $f(0) = 1$. Once again, there is a sudden change in output value at $x = 0$; in this case, as the input varies from $x = 0$ to $x = 1$ the output increases from a minimum of $y = -1$ (possibly at $x = 0$; again, we can't tell for sure) and increases to a maximum output of $f(1) = 0$. Therefore, the trace which best fits is (B).

Problem 6. Boris places his goldfish, Natasha, in a cylindrical glass bowl having a radius of three inches, and containing twelve inches of water. Natasha is one-half inch long and one-eighth of an inch wide. Boris proceeds to watch Natasha swim around for five minutes.

- Let t represent the values of the quantity "The number of minutes passed since Boris started watching Natasha."
- Let x represent the values of the quantity "The horizontal distance between Natasha's center and the central axis of the bowl, measured in millimeters."
- Let h represent the values of the quantity "The vertical distance between Natasha's center and the bottom of the bowl, measured in millimeters."

Part (a). Does t vary in the interval $[3.75,3.76]$?

Yes; it is reasonable to assume that t varies in any subinterval of $0 \leq t \leq 5$.

Part (b). Does x vary in the interval $[1.4,1.45]$?

We know that values of represent horizontal positions that are inside the bowl; however, we need to worry about whether Natasha can physically fit in the space when her midpoint is located at an x–value in this interval. Since Natasha is one-half inch long and one-eighth of an inch wide, Natasha's midpoint is 0.25 inches from her nose and her tail; and Natasha's midpoint is 0.125 inches from either of her sides. Therefore, regardless of how Natasha is oriented, when x is in this specified interval, Natasha's body remains inside the bowl.

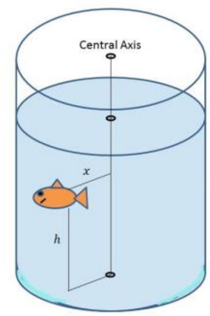
Part (c). Does x vary in the interval $[2.94, 2.95]$? No; see the reasoning from Part (b).

Problem 7. The diagram below shows Natasha in her bowl.

Part (a). Is the variable h a function of the variable t in the interval $[0,5]$?

Part (b). Is the variable t a function of the variable h in the interval $[0,12]$?

Part (c). Is the variable h a function of the variable x in the interval $[0,3]$?

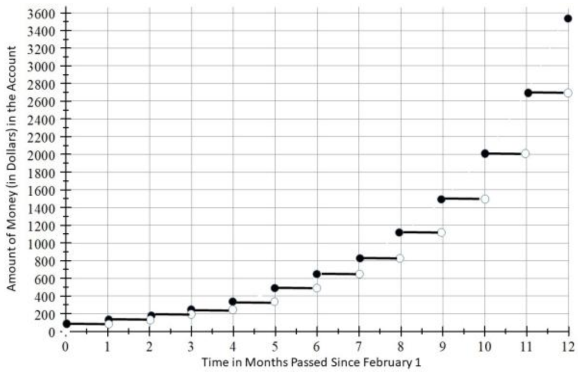


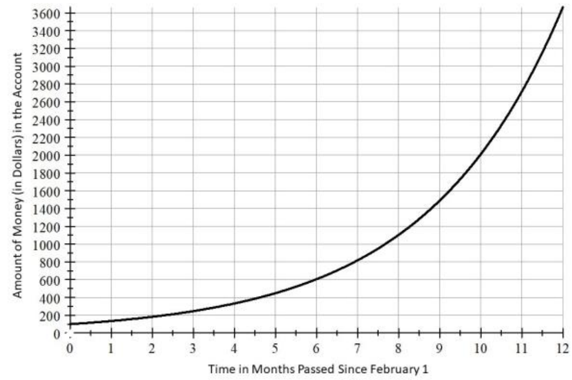
At any given time value, Natasha can only be in one location in the bowl; hence, h is a function of t in the interval $[0,5]$. However, since there are no restrictions on the manner in which Natasha moves in the bowl, it is possible that there could be many times when she is at a given height above the bottom. Therefore, t is not a function of h. If Natasha were to swim through the water in a horizontal line, she would pass through many values of x while the value of h did not change. Therefore, the variable x is not a function of the variable h. Likewise, if Natasha were to swim through the water in a vertical line (admittedly harder to do), then she would pass through many values of h while the value of x did not change. Therefore, the variable h is also not a function of the variable x.

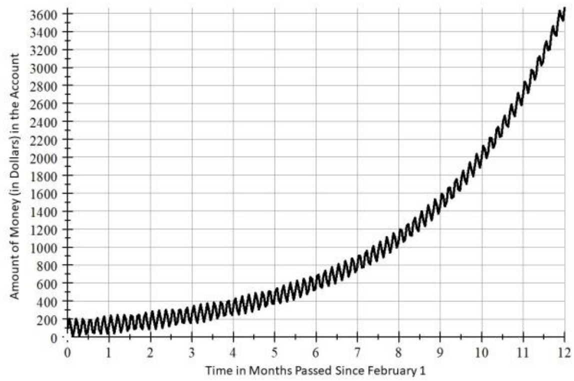
Problem 8. Suppose Terrance deposits \$100 into an account at the Incredible Bank of Murfreesboro (IBM) on February 1. The table below shows the amount of money in his account, rounded to the nearest dollar, at the end of each month passed since February 1.

Month	1	2	3	4	5	6	7	8	9	10	11	12
Amount in the Account	135	182	246	332	448	605	817	1,103	1,489	2,011	2,714	3,664

Let A represent measures of the quantity "the amount of money in the account, measured in dollars"; and let t represent measures of the quantity "the time passed since February 1, measured in months." The following traces represent potential graphs of the function $A = f(t)$ in the interval $0 \le t \le 12$. In each case, construct a reasonable scenario that would make the trace correct.







The leftmost graph above could represent the function $A = f(t)$ if the bank only pays interest to Terrance at the end of each month --- *and* Terrance does not withdraw or add money to his account. The rightmost graph above could represent the function $A = f(t)$ if the bank pays interest to Terrance on a *continuous* basis --- *and* Terrance does not withdraw money from his account. (Withdrawing would likely result in decreases in output.) The bottom graph could represent the function $A = f(t)$ if the bank pays interest to Terrance on a *continuous* basis --- *and* Terrance is withdrawing and adding small amounts of money to his account multiple times each month.

Problem 9. Suppose that p and w represent the measures of two varying quantities in a process, and suppose that

$$w = h(p) = 2p^2 - 1$$

Part (a). What does the letter h represent in this relationship?

The letter h is the name of the rule that assigns each value of p to exactly one value of w. This rule is represented by the formula that transforms each value of p into a value of w.

Part (b). What is the input variable for this relationship?

The variable p represents the input, and the variable w represents the output for the function h.

Part (d). Write down the formula for the expression

$$\frac{h(p+1)}{h(p)+1}$$

Observe that

$$\frac{h(p+1)}{h(p)+1} = \frac{2(p+1)^2 - 1}{(2p^2-1)+1} = \frac{2p^2+4p+1}{2p^2}$$

Problem 10. Suppose that r and s represent the measures of two varying quantities in a process, and suppose that

$$r = j(s) = \sqrt{2s+4}$$

Part (a). What does the letter j represent in this relationship?

The letter j is the name of the rule that assigns each value of s to exactly one value of r. This rule is represented by the formula that transforms each value of s into a value of r.

Part (b). What is the input variable for this relationship?

The variable s represents the input, and the variable r represents the output for the function j.

Part (d). What is the value of the expressions $j(6) \cdot j(6)$ and $j\big(j(6)\big)$?

Observe that

$$j(6) \cdot j(6) = \sqrt{2 \cdot 6 + 4} \cdot \sqrt{2 \cdot 6 + 4} = \sqrt{16} \cdot \sqrt{16} = 16$$

$$j\big(j(6)\big) = j\big(\sqrt{2 \cdot 6 + 4}\big) = j\big(\sqrt{16}\big) = j(4) = \sqrt{2 \cdot 4 + 4} = \sqrt{12}$$