The Chain Rule is one of the most useful procedures in calculus. In this investigation and the next, we will further explore the process of implicit differentiation introduced in the last investigation. We start by developing another important specific derivative formula.

Consider the function $y = f(x) = \sqrt[n]{x}$, where n is a positive integer. Note that this relationship can also be written

$$y^n = x$$

Differentiate both sides of this equation with respect to the variable x.

$$y^n = x \quad \Longrightarrow \quad \frac{d}{dx}[y^n] = \frac{d}{dx}[x]$$

$$\Longrightarrow \quad ny^{n-1} \cdot \frac{dy}{dx} = 1$$

$$\Longrightarrow \quad \frac{dy}{dx} = \frac{1}{ny^{n-1}}$$

Problem 1. Use the laws of exponents to show that if $y = \sqrt[n]{x}$, then $y^{n-1} = x^{1-1/n}$.

Problem 2. Use Problem 1 and the laws of exponents to show that

$$\frac{d}{dx}\left[\sqrt[n]{x}\right] = \frac{1}{n} \cdot x^{1/n-1}$$

A *rational number* is any ratio of integers. We can combine Problem 2 with the Chain Rule to determine the derivative formula for any rational power function.

Let m be any nonzero integer, and let n be any positive integer. Consider the function

$$y = f(x) = x^{m/n} = \sqrt[n]{x^m}$$

$$\frac{df}{dx} = \frac{d}{du}\left[\sqrt[n]{u}\right]\bigg|_{u=x^m} \cdot \frac{d}{dx}[x^m] \qquad \text{Let } u = x^m$$

$$= \frac{1}{n} \cdot (x^m)^{1/n-1} \cdot (mx^{m-1})$$

$$= \frac{m}{n} x^{(m/n-m)+(m-1)}$$

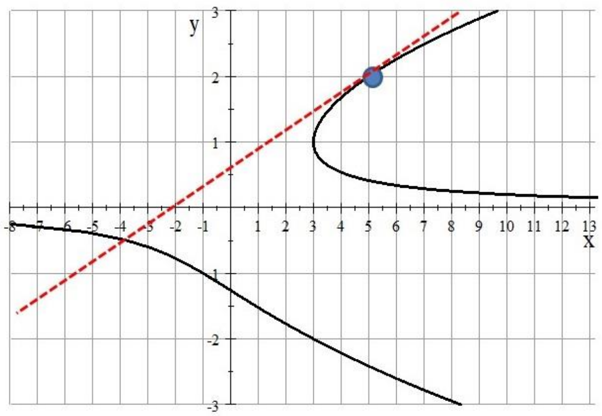
$$= \frac{m}{n} x^{m/n-1}$$

> ***Rational Power Formula***
>
> If r is any nonzero rational number, then $\frac{d}{dx}[x^r] = rx^{r-1}$.

Problem 3. If $y = f(x) = 2x^{2/3} - 4x^{3/4} + 3\log_4(x)$, then what is the value of $f'(8)$?

Implicit differentiation can be used to determine the instantaneous rate of change in one variable with respect to another for curves, even when one of the related variables cannot be expressed as a function of the other.

For example, consider the curve defined by the relation $xy = 2 + y^3$. The diagram on the following page shows the graph of this curve, along with the tangent line to the curve at the point $(5,2)$.



Problem 4. Estimate the slope of this tangent line.

Let's see how we could use the Chain Rule to determine the exact slope of the tangent line in Problem 8.

Problem 5. Consider the expression $xy = 2 + y^3$. Explain what is happening in each step below.

$xy = 2 + y^3 \quad \Rightarrow \quad \frac{d}{dx}[xy] = \frac{d}{dx}[2 + y^3]$ What just happened?

$\Rightarrow \quad x \cdot \frac{d}{dx}[y] + \frac{d}{dx}[x] \cdot y = \frac{d}{dx}[2] + \frac{d}{dx}[y^3]$ What just happened?

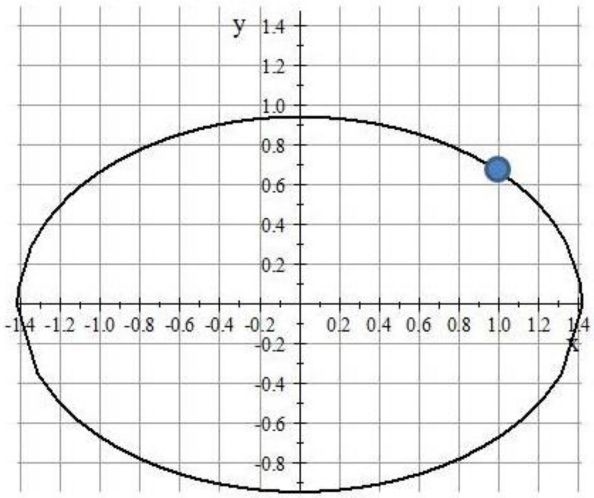
$\Rightarrow \quad x\frac{dy}{dx} + y = 0 + 3y^2\frac{dy}{dx}$ What just happened?

$\Rightarrow \quad y = (3y^2 - x)\frac{dy}{dx}$ What just happened?

$\Rightarrow \quad \frac{dy}{dx} = \frac{y}{3y^2 - x}$ What just happened?

Problem 6. At the point (5,2) on the curve, we know that $x = 5$ and $y = 2$. Use this information and your answer to Problem 10 to determine the exact value of the constant rate of change for the tangent line.

Problem 7. Consider the ellipse defined by the formula $4x^2 + 9y^2 = 8$. The graph of this ellipse is shown below.



Part (a). Carefully sketch the graph of the tangent line to the graph of the ellipse at the point $\left(1, \frac{2}{3}\right)$. Use your graph to estimate the slope of this tangent line.

Part (b). Find a formula for $\frac{dy}{dx}$ and use this formula to find the exact value of the constant rate of change for the tangent line at this point.

Homework.

Problem 1. Differentiate the function $h(x) = \tan(\log_5(x))$.

Problem 2. What is the second derivative of the function $y = h(x) = \ln(\cos(x))$?

Problem 3. Differentiate the function $y = f(x) = 3x^{2/3} + 2x^{-1/2} + \log_8(x)$.

Problem 4. Differentiate the function $y = f(x) = x^{7/8} + 5x^{9/5} + 10\log_7(x)$.

Problem 5. Consider the function $y = f(x) = \log_3(x^3 - 3x + 4)$. At what values of the input variable x will the tangent line to this function be horizontal?

Problem 6. Consider the curve defined by the formula $x^2 + y^2 - 4xy = -3$.

Part (a). Construct the formula for $\frac{dy}{dx}$.

Part (b). What is the formula for the line tangent to the graph of this curve at the point $(4\ ,\ 1)$?

Problem 7. Consider the curve defined by the formula $y\sin(x) = 4x\cos(y)$.

Part (a). Construct the formula for $\frac{dy}{dx}$.

Part (b). What is the instantaneous rate of change in the values of y with respect to the values of x at the point $(\pi/2\ ,\ 2\pi)$?

If α is any *real* number and b is any *positive* number, then we define the power expression b^α by

$$b^\alpha = e^{\alpha\cdot\ln(b)}$$

We make no attempt to define real powers of negative numbers.

Problem 8. Use the Chain Rule to establish the *Real Power Formula*:

- If α is a real number and $x > 0$, then the function $y = f(x) = x^\alpha$ is differentiable; in particular, we have $f'(x) = \alpha \cdot x^{\alpha-1}$.

The *arcsine* function $\theta = f(y) = \text{Arcsin}(y)$ serves as a *partial* inverse function for the sine function $y = g(\theta) = \sin(\theta)$. In particular, as long as $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$, we have the relationship

$$y = \sin(\theta) \quad \Leftrightarrow \quad \theta = \text{Arcsin}(y)$$

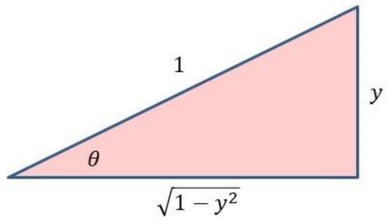
Likewise, the *arctangent* function $\alpha = f(y) = \text{Arctan}(y)$ serves as a *partial* inverse function for the tangent function $y = g(\alpha) = \tan(\alpha)$. In particular, as long as $-\frac{\pi}{2} < \alpha < \frac{\pi}{2}$, we have the relationship

$$y = \tan(\alpha) \quad \Leftrightarrow \quad \alpha = \text{Arctan}(y)$$

Problem 9. Use implicit differentiation and the relationship $y = \sin(\theta)$ and $y = \tan(\alpha)$ to show that

$$\frac{d}{dy}[\text{Arcsin}(y)] = \frac{d\theta}{dy} = \sec(\theta) \qquad \frac{d}{dy}[\text{Arctan}(y)] = \frac{d\alpha}{dy} = [\cos(\alpha)]^2$$

Problem 10. The restriction placed on the angle measure θ guarantees that the angle having measure θ is the vertex angle in a right triangle like the one shown below.



$$y = \sin(\theta) \quad \Rightarrow \quad \frac{y}{1} = \frac{\text{Side Opposite } \theta}{\text{Hypotenuse}}$$

Use this observation and the definition of the secant function to prove that

$$\frac{d}{dy}[\text{Arcsin}(y)] = \frac{1}{\sqrt{1 - y^2}}$$

Problem 11. Use the approach in Problem 10 to verify that

$$\frac{d}{dy}[\text{Arctan}(y)] = \frac{1}{1 + y^2}$$

You will need to construct a different triangle based on the right-triangle definition of the tangent function.

Answers to the Homework.

Problem 1. We have

$$h'(x) = \frac{\sec^2(\log_5(x))}{x\ln(5)}$$

Problem 2. We have

$$h'(x) = -\tan(x) \qquad h''(x) = \sec^2(x)$$

Problem 3. We have

$$f'(x) = 2x^{-1/3} - x^{-3/2} + \frac{1}{x\ln(8)}$$

Problem 4. We have

$$f'(x) = \frac{7}{8}x^{-1/8} + 9x^{4/5} + \frac{10}{x\ln(7)}$$

Problem 5. We have

$$f'(x) = \frac{3x^2 - 3}{(x^3 - 3x + 4)\ln(3)}$$

$$f'(x) = 0 \quad \Longrightarrow \quad 3x^2 - 3 = 0 \quad \Longrightarrow \quad x = \pm 1$$

Problem 6.

Part (a). We have

$$x^2 + y^2 - 4xy = -3 \;\Longrightarrow\; \frac{d}{dx}[x^2 + y^2 - 4xy] = \frac{d}{dx}[3]$$

$$\Longrightarrow\; \frac{d}{dx}[x^2] + \frac{d}{dx}[y^2] - 4\left(\frac{d}{dx}[x]y + x\frac{d}{dx}[y]\right) = \frac{d}{dx}[3]$$

$$\Longrightarrow\; 2x + 2y\frac{dy}{dx} - 4y - 4x\frac{dy}{dx} = 0$$

$$\Longrightarrow\; 2x - 4y = (4x - 2y)\frac{dy}{dx}$$

$$\Longrightarrow\; \frac{x - 2y}{2x - y} = \frac{dy}{dx}$$

Part (b). At the point $(4,1)$, we know the slope of the tangent line will be

$$\left.\frac{dy}{dx}\right|_{(x,y)=(4,1)} = \frac{4 - 2(1)}{2(4) - 1} = \frac{2}{7}$$

The formula for the tangent line will therefore be $y = \frac{2}{7}(x - 4) + 1$.

Problem 7.

Part (a). We have

$$y\sin(x) = 4x\cos(y) \;\Longrightarrow\; \frac{d}{dx}[y\sin(x)] = \frac{d}{dx}[4x\cos(y)]$$

$$\Longrightarrow\; \frac{d}{dx}[y]\sin(x) + y\frac{d}{dx}[\sin(x)] = 4\left(\frac{d}{dx}[x]\cos(y) + x\frac{d}{dx}[\cos(y)]\right)$$

$$\Longrightarrow\; \sin(x)\frac{dy}{dx} + y\cos(x) = 4\cos(y) - x\sin(y)\frac{dy}{dx}$$

$$\Longrightarrow\; \big(\sin(x) + 4x\sin(y)\big)\frac{dy}{dx} = 4\cos(y) - y\cos(x)$$

$$\Longrightarrow\; \frac{dy}{dx} = \frac{4\cos(y) - y\cos(x)}{\sin(x) + 4x\sin(y)}$$

Part (b). What is the instantaneous rate of change in the values of y with respect to the values of x at the point $(\pi/2\,,\,2\pi)$?

$$\left.\frac{dy}{dx}\right|_{(x,y)=(\pi/2,2\pi)} = \frac{4\cos(2\pi) - (2\pi)\cos(\pi/2)}{\sin(\pi/2) + 4\left(\frac{\pi}{2}\right)\sin(2\pi)} = 4$$

Problem 8. Observe

$$\frac{df}{dx} = \frac{d}{dx}\left[e^{\alpha\cdot\ln(x)}\right] = \left.\frac{d}{du}[e^u]\right|_{u=\alpha\cdot\ln(x)} \cdot \frac{d}{dx}[\alpha\cdot\ln(x)] = e^{\alpha\cdot\ln(x)}\cdot\frac{\alpha}{x} = \alpha\cdot\frac{x^\alpha}{x} = \alpha\cdot x^{\alpha-1}$$

Problem 9. Observe

$$y = \sin(\theta) \quad\Longrightarrow\quad \frac{d}{dy}[y] = \frac{d}{dy}[\sin(\theta)] \quad\Longrightarrow\quad 1 = \cos(\theta)\frac{d\theta}{dy} \quad\Longrightarrow\quad \frac{1}{\cos(\theta)} = \frac{d\theta}{dy}$$

$$y = \tan(\alpha) \quad\Longrightarrow\quad \frac{d}{dy}[y] = \frac{d}{dy}[\tan(\alpha)] \quad\Longrightarrow\quad 1 = [\sec(\alpha)]^2\frac{d\alpha}{dy} \quad\Longrightarrow\quad \frac{1}{[\sec(\alpha)]^2} = \frac{d\alpha}{dy}$$

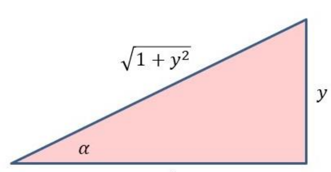
Problem 10. We know that the secant function can be expressed as a ratio of sides in this triangle; in particular, we know

$$\sec(\theta) = \frac{\text{Hypotenuse}}{\text{Side Adjacent to }\theta} = \frac{1}{\sqrt{1-y^2}}$$

Problem 11. The right-triangle definition of the tangent function tells us that

$$y = \tan(\alpha) \quad\Longrightarrow\quad \frac{y}{1} = \frac{\text{Side Opposite }\alpha}{\text{Side Adjacent to }\alpha}$$

Based on this information, the right triangle we should consider would be



Using this triangle, it is easy to see that

$$[\cos(\alpha)]^2 = \left(\frac{\text{Side Adjacent to }\alpha}{\text{Hypotenuse}}\right)^2 = \frac{1}{1+y^2}$$