In this investigation, we will look more closely at Euler's Method. This exploration will give rise to one of the most important processes in calculus. To introduce the concept, let's consider the following problem.

Consider the function $r = f(x) = 2x - \sqrt{x}$ in the interval $1 \le x \le 4$. Let $y = F(x)$ be the antiderivative for the function f that passes through the point $(1,2)$.

Problem 1. First off, we can determine the antiderivative that passes through the point $(1,2)$ using the antidifferentiation techniques we developed in Investigations 12 and 15.

Part (a). Compute the general antiderivative for the function f.

$$\int (2x - \sqrt{x})dx =$$

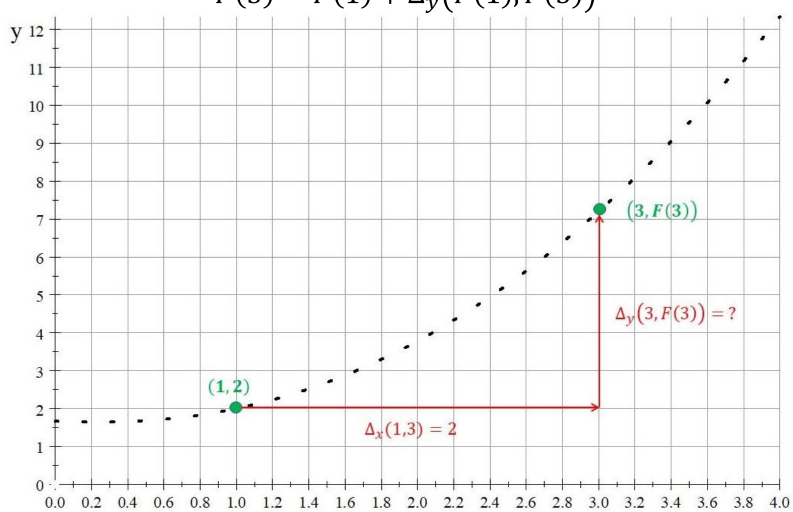
Part (b). Use the fact that we want $F(1) = 2$ to determine the value we need to use for the constant C in the general antiderivative formula.

$F(x) =$

We will use this exact formula for comparison purposes later; but now, suppose we *don't* know a formula for the function F. How could we go about finding the *exact* value of $F(3)$? (Or any other output value for that matter.)

We are assuming $F(1) = 2$, and we want to determine the value of $F(3)$. In order to do this, we need to know the *output change* in F as the values of x vary from $x = 1$ to $x = 3$. That is, we need a way to determine $F(3) - F(1) = \Delta_y\big(F(1), F(3)\big)$. If we could determine this change, then it follows that

$$F(3) = F(1) + \Delta_y\big(F(1), F(3)\big)$$



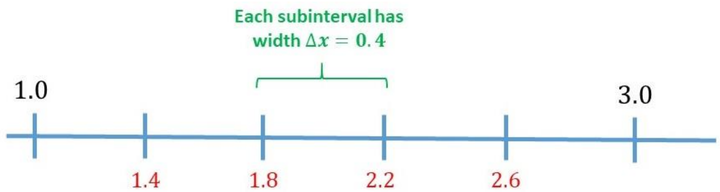
How could we determine the *exact* value of this change? Let's start by using Euler's Method to construct an *approximation*.

For concreteness, let's divide this interval into five equal-sized "steps". Each "step" will be a subinterval whose width is

$$\Delta x = \frac{3-1}{5} = 0.40$$

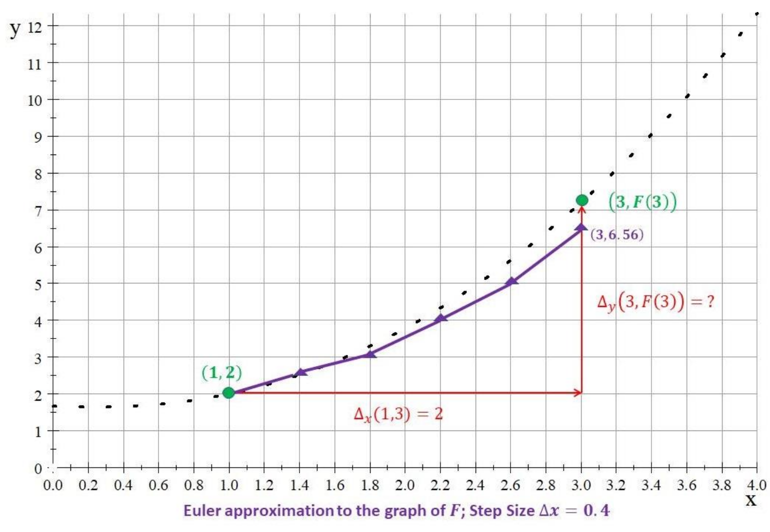
The endpoints of these "steps" give us input values we can use in Euler's Method:

$$a_0 = 1 \quad a_1 = 1.4 \quad a_2 = 1.8 \quad a_3 = 2.2 \quad a_4 = 2.6 \quad a_5 = 3.0$$



Problem 2. Use these input values to create a sequence of approximate output values for F:

- $F(1.4) \approx F(a_0) + F'(a_0) \cdot \Delta x =$
- $F(1.8) \approx F(a_1) + F'(a_1) \cdot \Delta x =$
- $F(2.2) \approx F(a_2) + F'(a_2) \cdot \Delta x =$
- $F(2.6) \approx F(a_3) + F'(a_3) \cdot \Delta x =$
- $F(3.0) \approx F(a_4) + F'(a_4) \cdot \Delta x =$



Let's take a closer look at these computations, because they contain important information about the output change in the function F.

When we apply Euler's Method to estimate $F(3)$ with a certain number of "steps" from the known value $F(1)$ the estimates actually build upon one another. For example, when we use five "steps", we have

- $F(a_0) = 2$
- $F(a_1) \approx F(a_0) + f(a_0) \cdot 0.4 \quad \Longrightarrow \quad F(a_1) - F(a_0) \approx f(a_0) \cdot 0.4$

Now, think about the computation that gave us the approximation for $F(a_2)$. Notice that

$$F(a_2) \approx F(a_1) + f(a_1) \cdot 0.4 \quad \Longrightarrow \quad F(a_2) \approx [F(a_0) + f(a_0) \cdot 0.4] + f(a_1) \cdot 0.4$$

$$\Longrightarrow \quad F(a_2) - F(a_0) \approx f(a_0) \cdot 0.4 + f(a_1) \cdot 0.4$$

Problem 3. Use this line of reasoning to determine a formula that approximates the output change $F(a_3) - F(a_0)$.

Problem 4. Based on your work in Problem 1, how could we approximate $F(3) - F(1)$ using the "steps" $x = a_0$ through $x = a_5$?

Problem 5. What happens if we try a smaller step-size on the input interval $1 \leq x \leq 3$? For example, suppose we use ten "steps" instead of five.

Part (a). In this case, what is the value of Δx?

Part (b). This choice for Δx gives us total of eleven input values where we can obtain approximations for the output of the mystery function F. What are these input values?

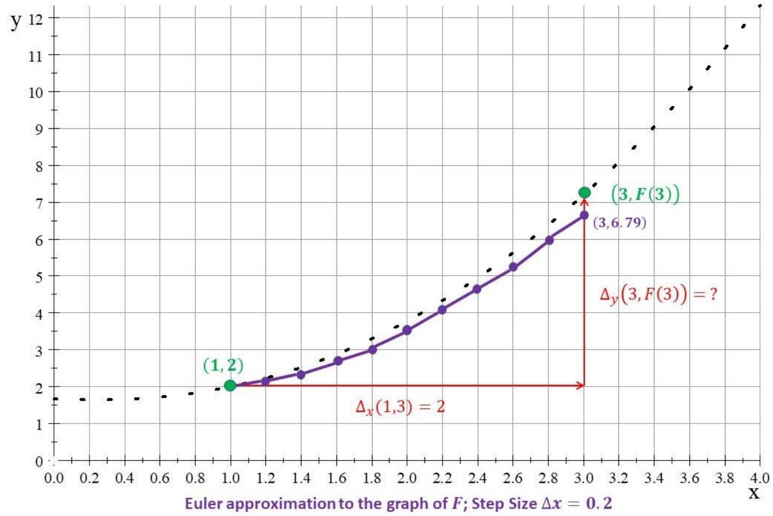
$a_0 =$ _____ $a_1 =$ ______ $a_2 =$ ______ $a_3 =$ ______ $a_4 =$ ______ $a_5 =$ ______

$a_6 =$ ______ $a_7 =$ ______ $a_8 =$ ______ $a_9 =$ ______ $a_{10} =$ ______

Problem 6. Use your "steps" and the assumption that $F(1) = 2$ to approximate the following output changes.

- $F(a_1) - F(1) \approx$
- $F(a_5) - F(1) \approx$
- $F(a_9) - F(1) \approx$
- $F(3) - F(1) \approx$

Note that the Euler approximation to the graph of F is slightly more accurate when we use ten "steps" instead of five. Consequently, the corresponding approximation to $F(3) - F(1)$ is also slightly more accurate.



Euler approximation to the graph of F; Step Size $\Delta x = 0.2$

> ***Euler Sums***
>
> Let f be a function in an interval $a \le x \le b$, and suppose that F is the antiderivative for f passing through the point $(a, F(a))$. If we divide the interval $a \le x \le b$ into n subintervals of equal width Δx, then
>
> $$F(b) - F(a) \approx f(a_0) \cdot \Delta x + f(a_1) \cdot \Delta x + \cdots + f(a_{n-1}) \cdot \Delta x = \sum_{j=0}^{n-1} f(a_j) \cdot \Delta x$$
>
> In this formula, we understand $a_j = a + j \cdot \Delta x$. This approximation for the output change is called an *Euler sum*.

In Problem 1, you showed that

$$F(x) = \frac{5}{3} + x^2 - \frac{2}{3}x\sqrt{x}$$

Problem 7. Consider this exact formula for the function F.

Part (a). What is the actual value of $F(3) - F(1)$ accurate to six decimal places?

Part (b). Use the following Euler sums to approximate $F(3) - F(1)$. What do you notice?

$$\sum_{j=0}^{4} f(a_j) \cdot \Delta x \qquad \sum_{j=0}^{9} f(a_j) \cdot \Delta x \qquad \sum_{j=0}^{15} f(a_j) \cdot \Delta x$$

Here are Euler sum approximations to the output change $F(3) - F(1)$ for much smaller "step" sizes.

$$\sum_{j=0}^{39} f(a_j) \cdot \Delta x \approx 5.12092 \qquad \sum_{j=0}^{99} f(a_j) \cdot \Delta x \approx 5.16989$$

It would seem that the Euler sums are slowly converging on the actual value of the output change for the function F as the number of "steps" used grows larger. This is indeed the case, at least with some restrictions on the function f.

First Fundamental Theorem of Calculus

Suppose that f is a continuous function in an interval $a \le x \le b$. If F is any antiderivative for f in this interval, then

$$F(b) - F(a) = \lim_{n\to+\infty} \sum_{j=0}^{n-1} f(a_j) \cdot \Delta x$$

Here, for each positive integer n, we understand that $\Delta x = \frac{b-a}{n}$, and we understand that each input value $a_j = a + j \cdot \Delta x$.

Problem 8. Consider the function $f(x) = xe^x$ in the interval $2 \le x \le 6$.

Part (a). Evaluate the expression $\sum_{j=0}^{7} f(a_j) \cdot \Delta x$.

$a_0 =$ _____ $a_1 =$ ______ $a_2 =$ ______ $a_3 =$ ______ $a_4 =$ ______

$a_5 =$ ______ $a_6 =$ ______ $a_7 =$ ______ $a_8 =$ ______

$$\sum_{j=0}^{7} f(a_j) \cdot \Delta x \approx$$

Part (b). Suppose we know that $F(2) = 3$. What would be the approximate value of $F(5)$ based on your computations?

Part (c). Suppose instead that we know $F(5) = 1$. What would be the approximate value of $F(2)$ based on your calculations?

It turns out that the limit of Euler sums used to compute the output change in the antiderivative functions F for a function f has many important uses of its own --- so much so that a special notation has been introduced simply to represent this limiting process.

If $r = f(x)$ is a continuous function in the input interval $a \le x \le b$, then it is customary to let

$$\int_{x=a}^{x=b} f(x)dx \text{ represent the end result of the limiting process } \lim_{n\to+\infty} \sum_{j=0}^{n-1} f(a_j) \cdot \Delta x$$

The end result of this process is called the *definite integral* for the function f over the interval $a \le x \le b$.

First Corollary of the FTC

Suppose that f is a continuous function in an interval $a \le x \le b$. If F is any antiderivative for the function f, then

$$\int_{x=a}^{x=b} f(x)dx = F(b) - F(a)$$

Problem 9. In the homework for Investigation 14, you showed that $F(x) = x^{-1}\sin(x)$ serves as one antiderivative for the function

$$f(x) = \frac{x\cos(x) - \sin(x)}{x^2}$$

Use this fact to evaluate the definite integral

$$\int_{x=\pi/2}^{x=2\pi} \left(\frac{x\cos(x) - \sin(x)}{x^2}\right) dx$$

Problem 10. Consider the function $f(x) = x^2 - 2x + 4$.

Part (a). Construct one antiderivative for the function f.

Part (b). Use your result from Part (a) to evaluate $\int_{x=-1}^{x=2} (x^2 - 2x + 4)\, dx$.

Homework.

Problem 1. Consider the function $f(x) = 3x^2$. You know that $F(x) = 4 + x^3$ is one antiderivative for the function f. Note that $F(-1) = 3$; hence, the graph of this antiderivative passes through the point $(-1,3)$. Let n be a positive integer, and let

$$\Delta x = \frac{2-(-1)}{n} = \frac{3}{n}$$

Part (a). Evaluate the following Euler sums.

$$\sum_{j=0}^{3} f(a_j)\cdot\Delta x \qquad\qquad \sum_{j=0}^{10} f(a_j)\cdot\Delta x$$

Part (b). Use the results of Part (a) to obtain two approximations for $F(2)$. Compare these approximations with the actual value of $F(2)$.

Evaluate the following definite integrals.

(2) $\int_{x=1}^{x=3} \ln(x)\,dx$ (3) $\int_{t=1}^{t=4} \left(\frac{1}{\sqrt{t}} - \frac{2}{\sqrt[3]{t}}\right) dt$ (4) $\int_{y=0}^{y=5/2} \cos(y)\,dy$ (5) $\int_{u=-1}^{u=2} e^u\,du$

(6) $\int_{x=2}^{x=6} (2x - 6x^2 + 5)\,dx$ (7) $\int_{t=-3}^{t=2} (2\sin(t) + 4)\,dt$ (8) $\int_{y=0}^{y=10} \left(y - \sqrt[4]{y^3}\right) dy$ (9) $\int_{u=0}^{u=\pi/4} \sec(u)\tan(u)\,du$

Answers.

Problem 1.

Part (a). In the first sum, we know that $n = 4$; hence, we also know $\Delta x = 0.75$ This tells us

$$a_0 = -1 \quad a_1 = -0.25 \quad a_2 = 0.50 \quad a_3 = 1.25 \quad a_4 = 2$$

$$\sum_{j=0}^{3} f(a_j)\cdot\Delta x = f(a_0)\cdot\Delta x + f(a_1)\cdot\Delta x + f(a_2)\cdot\Delta x + f(a_3)\cdot\Delta x$$

$$= 0.75(3\cdot(-1)^2 + 3\cdot(-0.25)^2 + 3\cdot(0.50)^2 + 3\cdot(1.25)^2) = 6.46875$$

In the second sum, we know that $n = 11$; hence, we also know $\Delta x \approx 0.27273$. This tells us

$$a_0 = -1 \quad a_1 \approx -0.72727 \quad a_2 \approx -0.45454 \quad a_3 \approx -0.18181 \quad a_4 \approx 0.09092 \quad a_5 \approx 0.36365$$

$$a_6 \approx 0.63638 \quad a_7 \approx 0.90911 \quad a_8 \approx 1.18184 \quad a_9 \approx 1.45457 \quad a_{10} \approx 1.7273 \quad a_{11} = 2$$

$$\sum_{j=0}^{10} f(a_j)\cdot\Delta x \approx 7.15089$$

Part (b). The values obtained in Part (a) will be estimates to $F(2) - F(-1)$. Consequently,

$$F(2) \approx F(-1) + \sum_{j=0}^{3} f(a_j) \cdot \Delta x = 3 + 6.46875 = 9.46875$$

$$F(2) \approx F(-1) + \sum_{j=0}^{10} f(a_j) \cdot \Delta x \approx 3 + 7.15089 = 10.15089$$

Notice that the second estimate is closer to the actual value $F(2) = 11$.

Problem 2. Since one antiderivative for $f(x) = \ln(x)$ is the function $F(x) = x\ln(x) - x$, we know

$$\int_1^3 \ln(x)\,dx = F(3) - F(1) = [3\ln(3) - 3] - [1\ln(1) - 1] = 3\ln(3) - 2$$

Problem 3. Observe that

$$\int \left(\frac{1}{\sqrt{x}} - \frac{2}{\sqrt[3]{x}}\right) dx = \int x^{-1/2}\,dx - 2\int x^{-1/3}\,dx = 2\sqrt{x} - 3\sqrt[3]{x^2} + C$$

We may use any constant C we wish, so let's let $C = -1$. We know

$$\int_{-1}^{4} \left(\frac{1}{\sqrt{x}} - \frac{2}{\sqrt[3]{x}}\right) dx = \left[2\sqrt{4} - 3\sqrt[3]{16} - 1\right] - \left[2\sqrt{1} - 3\sqrt[3]{1} - 1\right] = 5 - 6\sqrt[3]{2}$$

Problem 4. Observe that

$$\int \cos(x)\,dx = \sin(x) + C$$

We may use any constant C we wish, so let's let $C = 0$. We know

$$\int_0^{5/2} \cos(x)\,dx = \left[\sin\left(\frac{5}{2}\right)\right] - [\sin(0)] = \sin\left(\frac{5}{2}\right)$$

Problem 5. We know that $\frac{d}{du}[e^u] = e^u$, so we also know that

$$\int e^u\,du = e^u + C$$

We may use any constant C we wish, so let's let $= \pi$. We know

$$\int_{-1}^{2} e^u\,du = [e^2 + \pi] - [e^{-1} + \pi] = e^2 - \frac{1}{e}$$

Problem 6. Observe that

$$\int (2x - 6x^2 + 5)dx = 2\int x\,dx - 6\int x^2\,dx + \int 5\,dx = x^2 - 2x^3 + 5x + C$$

We may use any constant C we wish, so let's let $C = 0$ .Therefore, we know

$$\int_2^6 (2x - 6x^2 + 5)\,dx = [6^2 - 2(6)^3 + 5(6)] - [2^2 - 2(2)^3 + 5(2)] = 32$$

Problem 7. Observe that

$$\int (2\sin(x) + 4)\,dx = 2\int \sin(x)\,dx + \int 4dx \;= 4x - 2\cos(x) + C$$

We may use any constant C we wish, so let's let $= 4$. .

$$\int_{-3}^{2} (2\sin(x) + 4)\,dx = [8 - 2\cos(2) + 4] - [-12 - 2\cos(-3) + 4] = 20 - 2(\cos(2) - \cos(-3))$$

Problem 8. Observe that

$$\int \left(x - \sqrt[4]{x^3}\right) dx = \int x\,dx - \int x^{3/4}\,dx \;= \frac{x}{2} - \frac{4}{7}\sqrt[4]{x^7} + C$$

Let $C = 0$. We know

$$\int_0^{10} \left(x - \sqrt[4]{x^3}\right) dx = \left[5 - \frac{4}{7}\sqrt[4]{10^7}\right] - [0] = 5 - \frac{40}{7}\sqrt[4]{1000}$$

Problem 9. We know that $F(x) = \sec(x) - 5$ is one antiderivative for $f(x) = \sec(x)\tan(x)$; therefore, we also know

$$\int_0^{\pi/4} \sec(x)\tan(x)\,dx = \left[\sec\left(\frac{\pi}{4}\right) - -5\right] - [\sec(0) - 5] = \sqrt{2} - 1$$