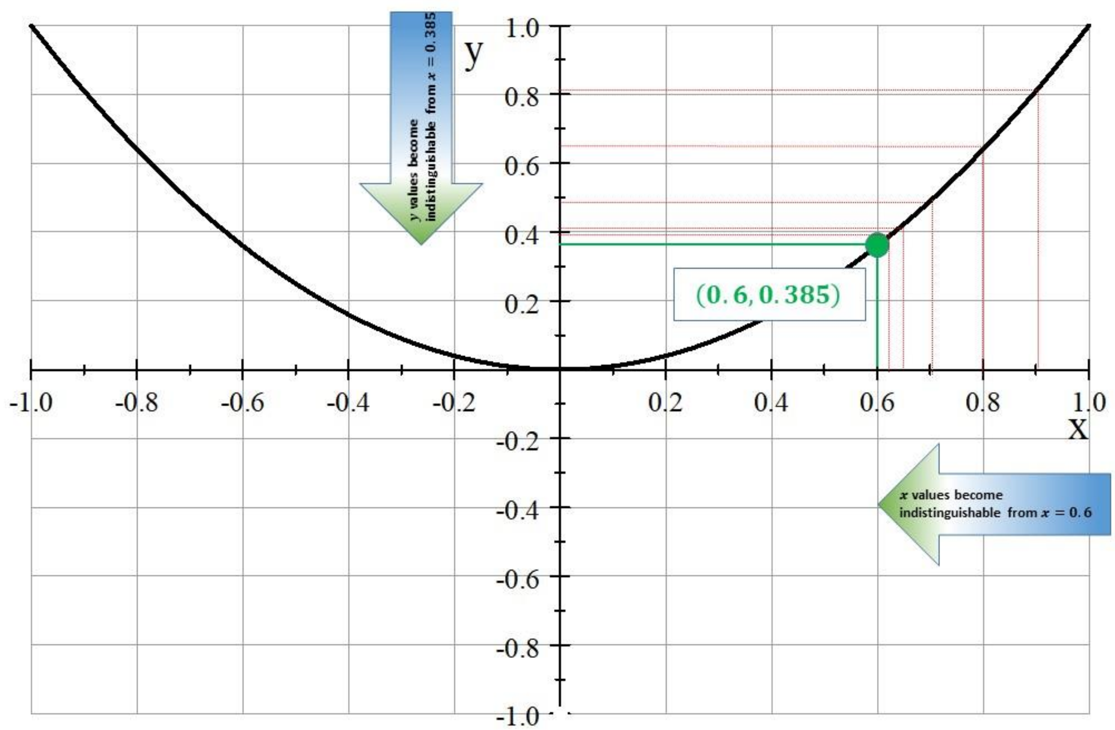
Suppose that $x$ and $y$ represent the values of two varying quantities, and suppose that $y = f(x)$. The assumption that $y$ is a function of $x$ means there is a dynamic relationship between the values of $y$ and the values of $x$. When the value of $x$ changes, the value of $y$ is forced to change as well. We sometimes say that the values of $y$ *covary* with the values of $x$ when there is a functional relationship between them.

***Continuity at an Input Value***

Suppose that $y = f(x)$ is a function in an interval $a \leq x \leq b$. We say that $f$ is *continuous* at an input value $x = c$ in this interval provided

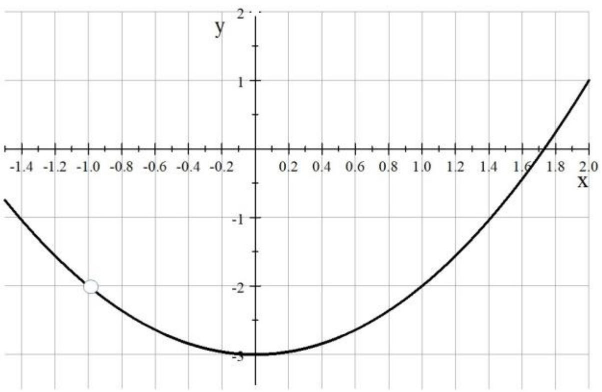
1. The output value $f(c)$ exists.
2. In the process of allowing the input values $x$ to become indistinguishable from $x = c$, the corresponding output values $f(x)$ also become indistinguishable from $y = f(c)$.

When the function $f$ is NOT continuous at an input value, we say that $f$ has a *discontinuity* at that input value.

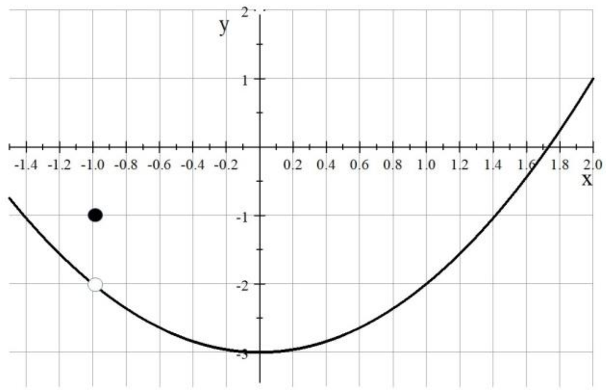



**Problem 1.** What do you think it means to say "the values of $f(x)$ *become indistinguishable* from $f(a)$"?

**Problem 2.** The diagrams below show the graph of two functions, $y = f(x)$ and $y = g(x)$. Neither function is continuous at the input value $x = -1$. Which criterion of the definition is violated in each case?

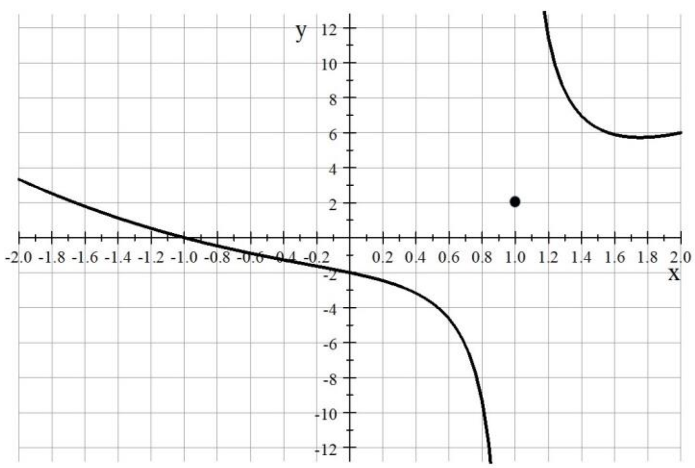


Graph of $f$

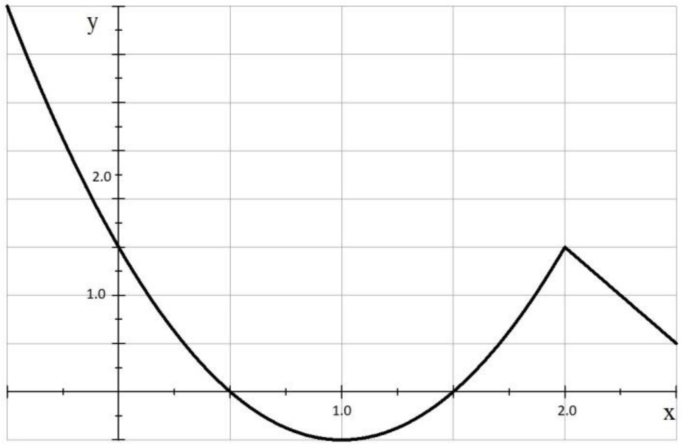


Graph of $g$

**Problem 3.** Consider the function $y = f(x)$ in the interval $-2 \leq x \leq 2$ whose graph is shown below. Do you think this function is continuous at the input value $x = 1$? Explain your thinking in terms of the definition.



**Problem 4.** Consider the function $y = f(x)$ in the interval whose graph is shown below. Do you think this function is continuous at the input value $x = 2$? Explain your thinking in terms of the definition.




**Problem 5.** Consider the function $y = f(x)$ defined by the following formula in the interval $-1 \le x \le 1$.

$$f(x) = \begin{cases} 0 & \text{if } x = 0 \\ \sin\left(\dfrac{\pi}{x}\right) & \text{if } x \ne 0 \end{cases}$$

**Part (a).** Make sure your graphing calculator is in radian mode and use it to fill in the following tables.

| Value of $x$ | $-1.0000$ | $-0.30000$ | $-0.01000$ | $-0.00300$ | $-0.00010$ | $-0.00003$ |
|---|---|---|---|---|---|---|
| Value of $f(x)$ | | | | | | |

| Value of $x$ | $-1.0000$ | $-0.75000$ | $-0.075000$ | $-0.00750$ | $-0.00075$ | $-0.000075$ |
|---|---|---|---|---|---|---|
| Value of $f(x)$ | | | | | | |

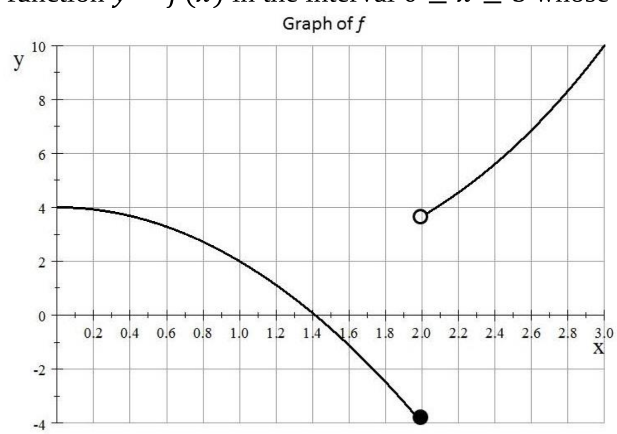
| Value of $x$ | $1.0000$ | $-0.75000$ | $0.066000$ | $-0.00750$ | $0.00066$ | $-0.000075$ |
|---|---|---|---|---|---|---|
| Value of $f(x)$ | | | | | | |

| Value of $x$ | $1.0000$ | $0.57000$ | $0.022000$ | $0.00830$ | $0.00015$ | $0.000042$ |
|---|---|---|---|---|---|---|
| Value of $f(x)$ | | | | | | |

Based on these tables, do you think that "the values of $f(x)$ become indistinguishable from $f(0)$" as "the input values $x$ become indistinguishable from $x = 0$"? Explain.

> ***One-Sided Limiting Processes***
>
> Suppose that $y = f(x)$ is a function, and suppose that $L$ is a real number. When we write
>
> $$\lim_{\substack{x\to a \\ \text{Keeping } x>a}} f(x) = L$$
>
> we mean that the values of $f(x)$ become indistinguishable from $y = L$ in the process of allowing values of $x$ to become indistinguishable from $x = a$ *from the right.* When we write
>
> $$\lim_{\substack{x\to a \\ \text{Keeping } x<a}} f(x) = L$$
>
> we mean that the values of $f(x)$ become indistinguishable from $y = L$ in the process of allowing values of $x$ to become indistinguishable from $x = a$ *from the left.*

**Problem 6.** Consider the function $y = f(x)$ in the interval $0 \le x \le 3$ whose graph is shown below.




**Part (a).** What do you think the value of $\lim_{\substack{x\to 2 \\ \text{Keeping } x<2}} f(x)$ should be? How did you decide?

**Part (b).** What do you think the value of $\lim_{\substack{x\to 2 \\ \text{Keeping } x>2}} f(x)$ should be? How did you decide?

**Part (c).** Do you think the function $f$ is continuous at the input value $x = 2$?

***Two-Sided Limiting Processes***

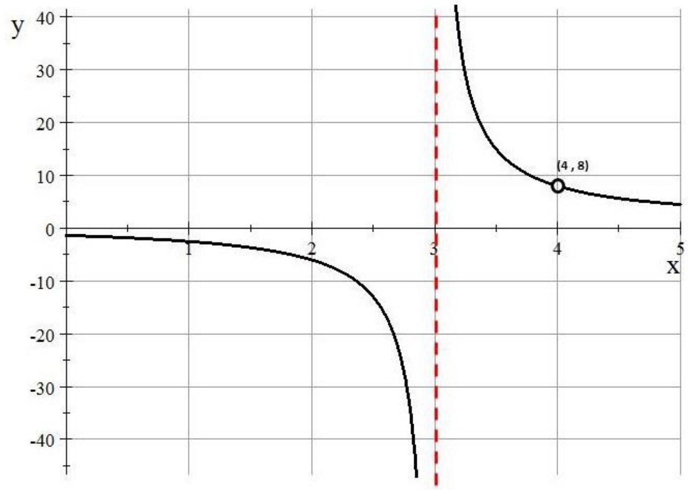
Suppose that $y = f(x)$ is a function, and suppose that $L$ is a real number. When we can show that

$$\lim_{\substack{x \to a \\ \text{Keeping } x<a}} f(x) = L = \lim_{\substack{x \to a \\ \text{Keeping } x>a}} f(x)$$

Then we say that the *two-sided* limiting process produces the *limiting value* $L$ as the values of $x$ become indistinguishable from $x = a$. In this case, we write

$$\lim_{x \to a} f(x) = L$$

**Problem 7.** Now, consider the function $y = f(x)$ whose graph is shown in the diagram below.



Graph of $f$

**Part (a).** Do you think there is a value for the two-sided limiting process $\lim_{x \to 4} f(x)$? Explain your thinking.

**Part (b).** Do you think there is a value for the limiting process $\lim_{\substack{x \to 3 \\ \text{Keeping } x>3}} f(x)$? Explain your thinking.

**Part (c).** Do you think there is a value for the limiting process $\lim_{\substack{x \to 3 \\ \text{Keeping } x<3}} f(x)$? Explain your thinking.

It is important to realize that the limiting processes in Parts (b) and (c) do not produce a real number --- the limit values do not exist. We often use the symbol "$\pm\infty$" to indicate that the output values of $f$ get more and more positive, or more and more negative. For example, we could write

$$\lim_{x\to 3^-} f(x) = -\infty \qquad\qquad \lim_{x\to 3^+} f(x) = +\infty$$

to describe the behavior we see from the graph of the function $f$ in Problem 7 above.

We can use the limit process to simplify the wording of our definition for continuity.

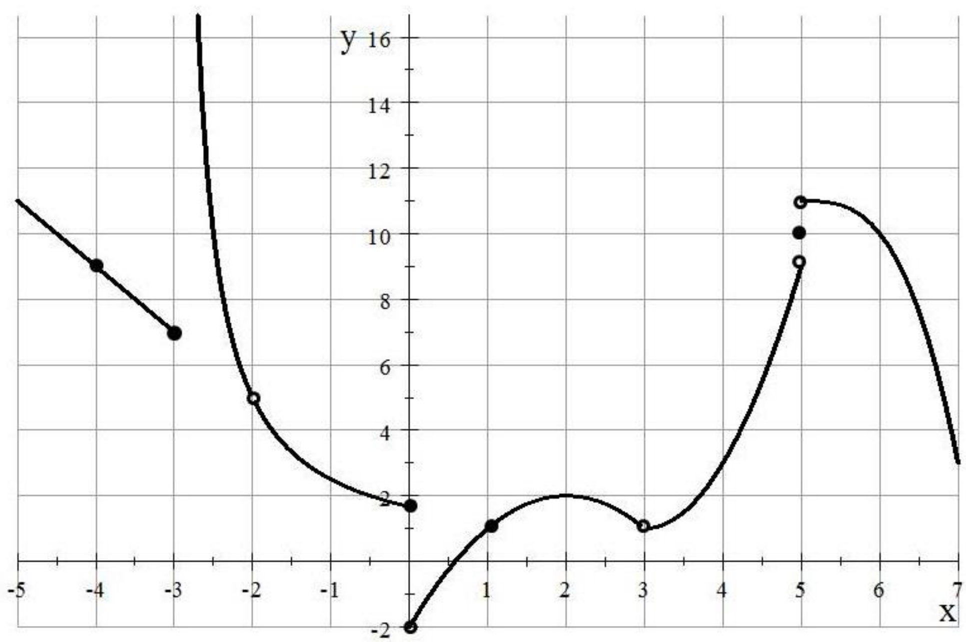
***Continuity at an Input Value***

Suppose that $y = f(x)$ is a function in an interval $a \le x \le b$. We say that the function $f$ is *continuous* at the input value $x = c$ in this interval provided the following conditions are met:

The output value $f(c)$ exists and $\lim_{x\to c} f(x) = f(c)$.

**Homework.**

**Problem 1.** Consider the graph of the function $y = f(x)$ shown in the diagram below.




Graph of $f$

Based on this graph, determine the value of the following limiting processes, if these values exist.

$(a)\ \lim_{x\to -4} f(x)$ $(b)\ \lim_{x\to -2} f(x)$ $(c)\ \lim_{\substack{x\to 5\\ \text{Keeping } x>5}} f(x)$ $(d)\ \lim_{\substack{x\to 0\\ \text{Keeping } x<0}} f(x)$

$(e)\ \lim_{\substack{x\to -3\\ \text{Keeping } x<-3}} f(x)$ $(f)\ \lim_{x\to 0} f(x)$ $(g)\ \lim_{\substack{x\to -3\\ \text{Keeping } x>-3}} f(x)$ $(h)\ \lim_{x\to 3} f(x)$

**Problem 2.** Let $y = f(x)$ be a function in the interval $a \le x \le b$ and let $x = c$ be an input value from this interval. Assume that the variable $h$ varies in the interval $0 < h \le b - c$. Do the following processes produce the same result?

$$\lim_{\substack{x\to c\\ \text{Keeping } x>c}} f(x) \qquad \lim_{h\to 0} f(c+h)$$

Salida says the following statement could be used to define continuity of a function at an input value:

- A function $y = f(x)$ is continuous at an input value $x = a$ provided letting the input values get closer and closer to $a$ causes the output values of $f$ get closer and closer to $y = f(a)$.

**Problem 3.** function $y = f(x)$ defined by the formula

$$f(x) = \begin{cases} 1.5 & \text{if} \quad x = 1 \\ 2 + |x-1| & \text{if} \quad x \neq 1 \end{cases}$$

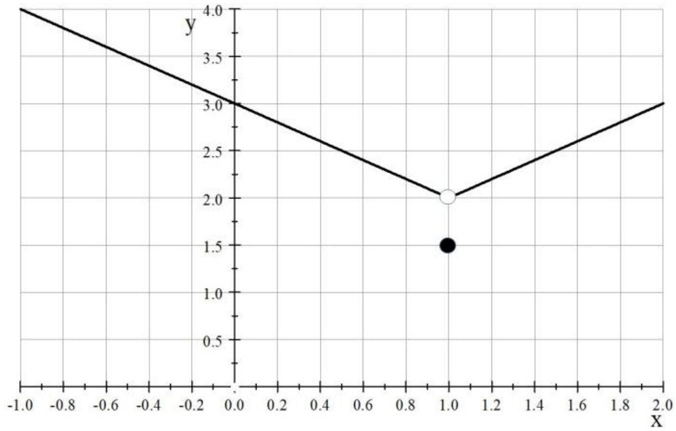
Use the formula to fill in the following tables.

| Value of $x$ | 0.0000 | 0.5000 | 0.9000 | 0.9900 | 0.9999 | 0.999999 |
|---|---|---|---|---|---|---|
| $f(x)$ | | | | | | |

| Value of $x$ | 2.0000 | 1.5000 | 1.0100 | 1.00100 | 1.00010 | 1.00001 |
|---|---|---|---|---|---|---|
| $f(x)$ | | | | | | |

Are the output values of $f$ getting closer and closer to the value of $y = f(1)$ as the values of $x$ get closer and closer to $x = 1$?

**Problem 4.** The diagram below shows the graph of the function $f$ from Problem 6.




Based on this graph, is the function $f$ continuous at the input value $x = 1$? Explain your answer.

**Answers to the Homework.**

**Problem 1.**

$(a)\ \lim_{x\to-4} f(x) = 9$ $\quad (b)\ \lim_{x\to-2} f(x) = 5$ $\quad (c)\ \lim_{\substack{x\to5 \\ \text{Keeping } x>5}} f(x) = 11$ $\quad (d)\ \lim_{\substack{x\to0 \\ \text{Keeping } x<0}} f(x) \approx 1.7$

$(e)\ \lim_{\substack{x\to-3 \\ \text{Keeping } x<-3}} f(x) = 7$ $\quad (f)\ \lim_{x\to0} f(x)$ DNE $\quad (g)\ \lim_{\substack{x\to-3 \\ \text{Keeping } x>-3}} f(x) = +\infty$ $\quad (h)\ \lim_{x\to3} f(x) = 1$

**Problem 2.** We are assuming that $h$ varies in the interval $0 < h \le b - c$. (We made this restriction to guarantee that $x = c + h$ stays in the input interval $a \le x \le b$.) We assume that $h$ can take on *all* values in the interval $(0, b - c]$ as it varies. Consequently, if $x$ is any input value in the interval $(c, b]$, there exists a value of $h$ such that $x = c + h$ --- namely $h = x - c$. Therefore, these two limiting processes must produce the same result (whatever that result may be).

**Problem 3.**

| Value of $x$ | 0.0000 | 0.5000 | 0.9000 | 0.9900 | 0.9999 | 0.999999 |
|---|---|---|---|---|---|---|
| $f(x)$ | 3 | 2.5 | 2.1 | 2.01 | 2.001 | 2.000001 |

| Value of $x$ | 2.0000 | 1.5000 | 1.0100 | 1.00100 | 1.00010 | 1.00001 |
|---|---|---|---|---|---|---|
| $f(x)$ | 3 | 2.5 | 2.01 | 2.001 | 2.0001 | 2.000001 |

Are the output values of $f$ getting closer and closer to the value of $y = f(1)$ as the values of $x$ get closer and closer to $x = 1$?

Yes; the difference between 1.5 and $f(x)$ gets smaller as the values of $x$ get closer and closer to $x = 1$.

**Problem 4.** No, the function is *not* continuous at $x = 1$. Based on the graph (as well as the tables above), we can see that

$$\lim_{x \to 2} f(x) = 2$$