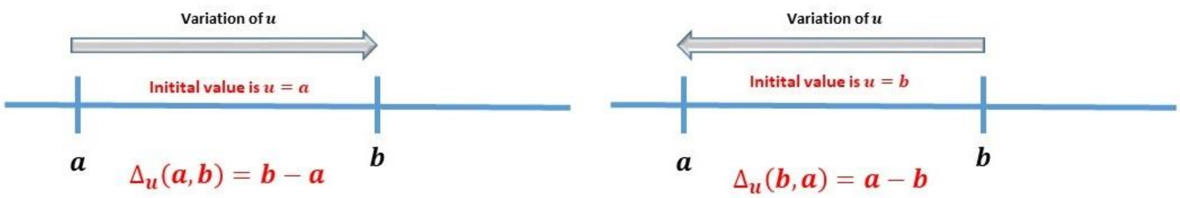
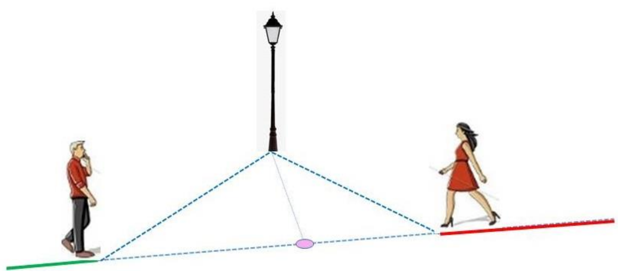
In the previous investigation, we introduced functions as *actions* that coordinate the variation of one variable with the variation of another variable in a specific way --- so that every value of the input variable is assigned to exactly one value of the output variable. Calculus is really the science of studying these actions.

> ***Change in a Variable***
>
> Suppose that u represents measures of a varying quantity in a process, and suppose u varies in an interval I. The *change* in u is defined to be the difference between the initial and final values of the variation that we are considering. In symbols, if $u = m$ is the initial value and $u = n$ is the final value in the variation, we let $\Delta_u(m, n)$ represent this change.



Roland waived to Taffy from opposite sides of a forty-foot wide parking lot. At the same moment, they started walking toward each other on an east-west line. Suppose there is a lamp post ten feet due north of a wad of chewing gum that was halfway between Roland and Taffy when they first saw each other.



Problem 1. Let a represent measures of the quantity "Taffy's distance *west* of her starting point, measured in feet."

Part (a). In the context of this problem, what is the meaning of $\Delta_a(4.72, 11.78)$? What is the value?

Part (b). Let b represent measures of the quantity "Taffy's distance *east* of the wad of gum, measured in feet." In the context of this problem, what is the meaning of $\Delta_b(15.312, -2.71)$? What is the value?

Problem 2. Let u represent measures of the quantity "Roland's distance *west* of the wad of gum, measured in feet." The variable u is a function of the variable b in the interval $[-20,20]$; let $u = f(b)$ denote the function action that that coordinates the values of u and b.

Part (a). Based on what we are told in Problem 1, what is the value of $f(20)$? Explain your thinking.

Part (b). Suppose that for *all* values of $b = m$ and $b = n$ that b takes on in the interval $[-20,20]$, we also know

$$\Delta_u\big(f(m), f(n)\big) = 0.75\Delta_b(m, n)$$

As b varies from $b = 20$ to $b = 12$, what is the corresponding change in u?

Part (c). Using Parts (a) and (b), what is the value of $f(12)$? Explain your strategy.

Part (d). As b varies from $b = 12$ to $b = 6$ feet, what is the corresponding change in u? What is the value of $f(6)$?

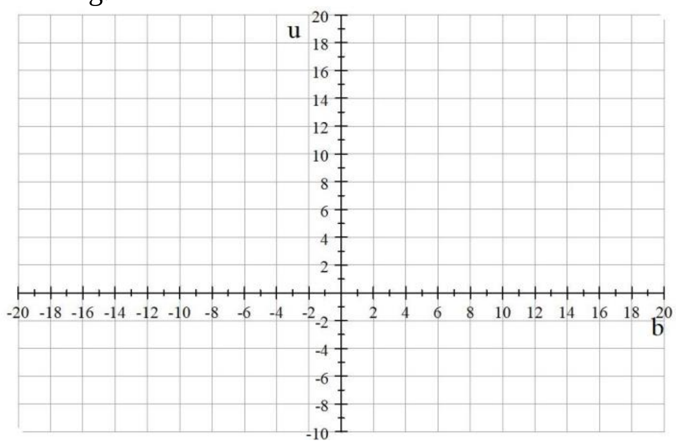
Part (e). What is the value of $f(0)$? Explain your strategy.

Part (f). Do Taffy and Roland meet at the wad of gum? Explain your thinking.

Problem 3. Consider the function $u = f(b)$ from Problem 2.

Part (a). On the grid provided, plot the ordered pairs $(20, f(20))$, $(12, f(12))$, $(6, f(6))$, and $(0, f(0))$.

Part (b). Sketch the trace that you think best represents the graph of the function f in the interval $[-20, 20]$. Explain your thinking.



Problem 4. In Problems 2 and 3, we assumed that $\Delta_u\big(f(m), f(n)\big) = 0.75\Delta_b(m, n)$ for *every* pair of input values $b = m$ and $b = n$ in the interval $-20 \le b \le 20$. Explain why this assumption tells us the following expressions are all correct algebraic formulas for f.

$$f(b) = 0.75(b - 20) + f(20) \qquad f(b) = 0.75\left(b + \frac{3}{2}\right) + f\left(-\frac{3}{2}\right) \qquad f(b) = 0.75(b - \pi) + f(\pi)$$

> ***Linear Function***
>
> Suppose x and y represent the measures of varying quantities in a process, and suppose $y = f(x)$ in an interval I. We say that f is a *linear function* provided there exists a constant m such that $\Delta_y\big(f(c), f(d)\big) = m\Delta_x(c, d)$ for *every* pair of input values $x = c$ and $x = d$ in the interval I.
>
> The number m is called the *constant rate of change* in y *with respect to* x.

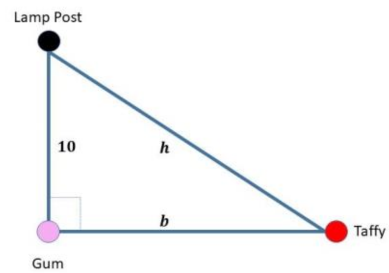
We refer to the constant m as a "rate of change" because this number is the ratio of two changes; in particular,

$$m = \frac{\Delta_y\big(f(c), f(d)\big)}{\Delta_x(c, d)} = \frac{f(d) - f(c)}{d - c}$$

Problem 5. Let h represent measures of the quantity "The line-of-sight distance between the lamp post and Taffy, measured in feet." (This variable represents the length of the dashed line between the lamp post and Taffy.) The variable h is a function of the variable b in the interval $[-20,20]$; let $h = G(b)$.

Part (a). Do you think that the function G is linear in the interval $[-20,20]$? Carefully explain your thinking.

Part (b). Use the diagram on the right to explain why the algebraic formula for G is given by $G(b) = \sqrt{100 + b^2}$.



Part (c). Fill in the table below. Based on this table, would you change your answer to Part (a)? Explain.

$\Delta_b(b_1, b_2)$	Value of b	Value of $G(b)$	$\Delta_h\big(G(b_1), G(b_2)\big)$
	20		
	15		
	10		
	5		

Homework

Problem 1. Paul was sitting on a park bench and decided to get up and go for a walk. When he passed a trash bin twelve feet from the bench, he started walking at a constant velocity toward a bus stop 250 feet from the bench. After passing the trash bin, Paul walked 52.8 feet in one eight-second time interval.

- Let t represent measures of the quantity "the time elapsed since Paul passed the trash bin, measured in seconds."
- Let z represent measures of the quantity "the distance in feet between Paul and the park bench."

Part (a). How far did Paul walk in 14 seconds? Does this value represent Paul's distance from the park bench?

Part (b). Does your answer to Part (a) depend on which 14-second interval we're talking about? Explain.

Part (c). How long did it take Paul to travel any 20-foot distance after he passed the trash bin?

Problem 2. Consider Paul and his walk. Let $t = g(z)$ represent the action that coordinates the values of t and z in the interval $12 \leq z \leq 250$ feet.

Part (a). As z varies from $z = 12$ to $z = 20$ feet, what is the corresponding change in t?

Part (b). What is the value of $g(20)$?

Part (c). As z varies from $z = 20$ to $z = 7\pi$ feet, what is the approximate corresponding change in t?

Part (d). What is the approximate value of $g(7\pi)$?

Part (e). What formula gives $\Delta_t\big(g(a), g(b)\big)$ in terms of $\Delta_z(a, b)$ for *any* pair $z = a$ and $z = b$ in the interval $[12{,}250]$?

Problem 3. Olivine is mixing up some Crystal Light drink mix. To avoid lumps, Olivine first pours eight ounces of water into the pitcher before adding the drink mix. This volume fills the pitcher to a depth of 2.25 inches. She adds the mix, then starts adding water to the pitcher at a constant rate. Whenever she adds 60 ounces of water, the depth of water in the pitcher will increase by 7.8 inches (until the pitcher is full).

- Let V represent measures of the quantity "the volume of the water in the pitcher, measured in ounces."
- Let y represent measures of the quantity "the depth of water *added* to the pitcher, measured in inches."

Suppose the pitcher is fifteen inches tall. Let $V = f(y)$ in the interval $0 \leq y \leq 12.75$ and assume that f is a linear function.

Part (a). What is the volume of water in the pitcher when $y = 0$ inches?

Part (b). As y varies from $y = 0$ to $y = 10.05$ inches, what is the corresponding change in V?

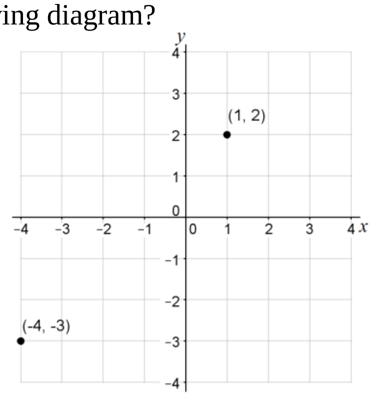
Part (c). What is the value of $f(10.05)$?

Part (d). As y varies from $y = 10.05$ inches to $y = 13$ inches, what is the corresponding change in V?

Part (e). What is the value of $f(13)$?

Part (f). What is the total volume of water the pitcher can hold?

Problem 4. Suppose x and y represent measures of varying quantities in a process, and suppose that $y = f(x)$ in the interval $[-4,4]$. If we know f is a linear function, what is the constant rate of change in y with respect to x based on the following diagram?



Problem 5. Suppose x and y represent measures of varying quantities in a process, and suppose that $y = f(x)$ is a linear function in an interval I. If m denotes the constant rate of change in y with respect to x, explain why we may conclude that $f(x) = m(x - a) + f(a)$ for *any* a in the interval I.[1]

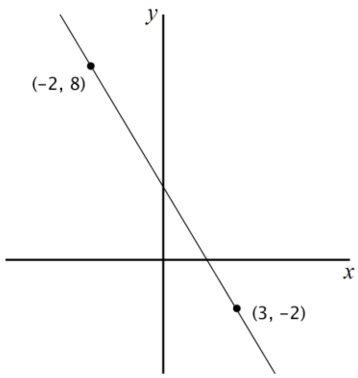
Problem 6. Suppose x and y represent measures of varying quantities in a process, and suppose that $y = f(x)$ is a linear function in the interval $-3 \leq x \leq 5$. If the constant rate of change in y with respect to x is $m = -\pi\sqrt{2}$ and we know $f(2.2) = \sqrt[3]{5}$, construct an algebraic formula for the function f.

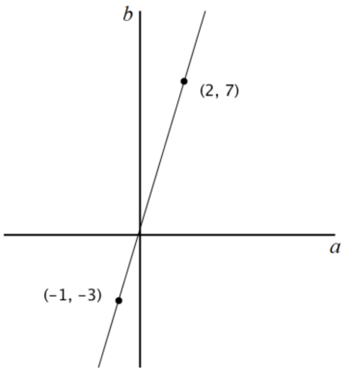
Problem 7. For each trace below, construct an algebraic formula that gives the values of one variable as a function of the values of the other variable. Use proper function notation.

a.

b.

[1] This formula is often called the *point-slope* representation for the linear function.





Answers to Homework.

Problem 1. Paul was sitting on a park bench and decided to get up and go for a walk. When he passed a trash bin twelve feet from the bench, he started walking at a constant velocity toward a bus stop 250 feet from the bench. After passing the trash bin, Paul walked 52.8 feet in one eight-second time interval.

- Let t represent measures of the quantity "the time elapsed since Paul passed the trash bin, measured in seconds."
- Let z represent measures of the quantity "the distance in feet between Paul and the park bench."

Part (a). How far did Paul walk in 14 seconds? Does this value represent Paul's distance from the park bench?

Let $z = f(t)$ represent the function action. Since Paul is assumed to be walking at a constant velocity, we know there exists some constant m such that $\Delta_z\big(f(a), f(b)\big) = m\Delta_t(a, b)$ for *any* time values $t = a$ and $t = b$. Furthermore, we are told that during one part of the walk, $\Delta_z\big(f(a), f(b)\big) = 52.8$ feet when $\Delta_t(a, b) = 8$ seconds. This tells us

$$m = \frac{\Delta_z\big(f(a), f(b)\big)}{\Delta_t(a, b)} = 6.6 \text{ feet per second}$$

Consequently, when $\Delta_s(a, b) = 14$ seconds, it follows that

$$\Delta_z\big(f(a), f(b)\big) = \left(6.6 \frac{\text{feet}}{\text{second}}\right) \cdot (14 \text{ seconds}) = 92.4 \text{ feet}$$

We are considering a *change* in time of fourteen seconds, but we are not told *which* 14-second change in time (in other words, we don't know the actual values of $t = a$ and $t = b$.) Whenever the value of the time variable increases $t = a$ to $t = a + 14$ seconds, we know that the value of the distance variable increases from $z = f(a)$ to $z = f(a) + 92.4$ feet. We cannot say more than this without additional information.

Part (b). Does your answer to Part (a) depend on which 14-second interval we're talking about? Explain.

No, it does not. See the explanation in Part (a).

Part (c). How long did it take Paul to travel any 20-foot distance after he passed the trash bin?

In this case, we are told that $\Delta_z\big(f(a), f(b)\big) = 20$ feet. Consequently, we know the corresponding change in time will be

$$\Delta_t(a,b) = \left(\frac{1}{6.6}\ \frac{\text{second}}{\text{foot}}\right)(20 \text{ feet}) \approx 3.03 \text{ seconds}$$

Problem 2. Consider Paul and his walk. Let $t = g(z)$ represent the action that coordinates the values of t and z in the interval $12 \le z \le 250$ feet.

Part (a). As z varies from $z = 12$ to $z = 20$ feet, what is the corresponding change in t?

We know that every change of 52.8 feet in the value of z corresponds to a change of eight seconds in the value of t; in other words, whenever $\Delta_z(u,v) = 52.8$ feet, we know $\Delta_t\big(g(u), g(v)\big) = 8$ seconds. The constant rate of change is therefore

$$n = \frac{\Delta_t\big(g(u), g(v)\big)}{\Delta_z(u,v)} \approx 0.152 \text{ seconds per foot}$$

Consequently, when $\Delta_z(u,v) = 8$ feet, we know $\Delta_t\big(g(u), g(v)\big) \approx 0.152 \cdot 8 = 1.216$ seconds.

Part (b). What is the value of $g(20)$?

It is critical to keep track of the reference points for our measurements. Note that $z = 0$ corresponds to Paul being at the park bench, while $z = 12$ corresponds to Paul being at the trash bin. However, we also know that $t = 0$ corresponds to Paul being at the trash bin. Therefore, we know $g(12) = 0$. Since it took 1.216 seconds for Paul to travel eight feet from the trash bin, we must conclude $g(20) = 1.216$.

Part (c). As z varies from $z = 20$ to $z = 7\pi$ feet, what is the approximate corresponding change in t?

We know $\Delta_z(20, 7\pi) \approx 1.9911$ feet. Hence, $\Delta_t\big(g(20), g(7\pi)\big) \approx 0.152 \cdot 1.9911 = 0.3026$ seconds.

Part (d). What is the approximate value of $g(7\pi)$?

$$g(7\pi) = g(20) + \Delta_t\big(g(20), g(7\pi)\big) \approx 1.216 + 0.3026 = 1.5186$$

Part (e). What formula gives $\Delta_t\big(g(a), g(b)\big)$ in terms of $\Delta_z(a,b)$ for *any* pair $z = a$ and $z = b$ in the interval $[12, 250]$?

$$\Delta_t\big(g(a), g(b)\big) \approx 0.152\Delta_z(a,b)$$

Problem 3. Olivine is mixing up some Crystal Light drink mix. To avoid lumps, Olivine first pours eight ounces of water into the pitcher before adding the drink mix. This volume fills the pitcher to a depth of 2.25 inches. She adds the mix, then starts adding water to the pitcher at a constant rate. Whenever she adds 60 ounces of water, the depth of water in the pitcher will increase by 7.8 inches (until the pitcher is full).

- Let V represent measures of the quantity "the volume of the water in the pitcher, measured in ounces."
- Let y represent measures of the quantity "the depth of water *added* to the pitcher, measured in inches."

Suppose the pitcher is fifteen inches tall. Let $V = f(y)$ in the interval $0 \leq y \leq 12.75$ and assume that f is a linear function.

Part (a). What is the volume of water in the pitcher when $y = 0$ inches? By assumption, $f(0) = 8$.

Part (b). As y varies from $y = 0$ to $y = 10.05$ inches, what is the corresponding change in V?

We know that when $\Delta_V\big(f(a), f(b)\big) = 60$ ounces, then $\Delta_y(a, b) = 7.8$ inches. The constant rate of change is therefore

$$m = \frac{\Delta_V\big(f(a), f(b)\big)}{\Delta_y(a, b)} = \frac{60 \text{ ounces}}{7.8 \text{ inches}} \approx 7.692 \text{ ounces per inch}$$

Thus, we know $\Delta_V\big(f(a), f(b)\big) \approx 7.692\Delta_y(a, b)$. Consequently, we know

$$\Delta_V\big(f(0), f(10.05)\big) \approx 7.692 \cdot 10.05 = 77.305 \text{ ounces}$$

Part (c). What is the value of $f(10.05)$?

$$f(10.05) = f(0) + \Delta_V\big(f(0), f(10.05)\big) \approx 8 + 77.305 = 85.305 \text{ ounces}$$

Part (d). As y varies from $y = 10.05$ inches to $y = 13$ inches, what is the corresponding change in V?

$$\Delta_V\big(f(10.05), f(13)\big) \approx 7.692 \cdot 2.95 = 22.691 \text{ ounces}$$

Part (e). What is the value of $f(13)$?

$$f(13) = f(10.05) + \Delta_V\big(f(10.05), f(13)\big) \approx 85.305 + 22.691 = 107.996 \text{ ounces}$$

Part (f). What is the total volume of water the pitcher can hold?

If we assume Olivine fills the pitcher to the very top, then the total volume is $f(15)$. Now, we know

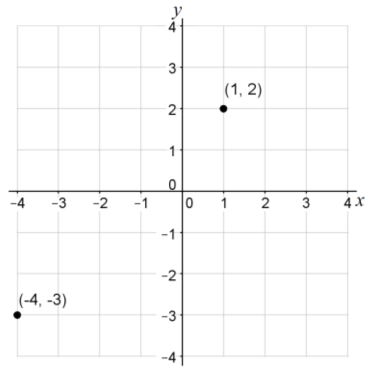
$$\Delta_V\big(f(13), f(15)\big) \approx 7.692 \cdot 2.00 = 15.384 \text{ ounces}$$

Hence, we also know $f(15) = f(13) + \Delta_V\big(f(13), f(15)\big) \approx 123.38$ ounces.

Problem 4. Suppose x and y represent measures of varying quantities in a process, and suppose that $y = f(x)$ in the interval $[-4,4]$. If we know f is a linear function, what is the constant rate of change in y with respect to x based on the following diagram?

Using the two specified points, we know

$$\Delta_y\big(f(-3), f(1)\big) = m\Delta_x(-3,1) \quad \Rightarrow \quad 1 - (-4) = m \cdot 4 \quad \Rightarrow \quad \frac{5}{4} = m$$



Problem 5. Suppose x and y represent measures of varying quantities in a process, and suppose that $y = f(x)$ is a linear function in an interval I. If m denotes the constant rate of change in y with respect to x, explain why we may conclude that $f(x) = m(x-a) + f(a)$ for *any* a in the interval I.[2]

First, note that by assumption, for *any* input values $x = a$ and $x = b$ in the interval I, we have

$$\Delta_y\big(f(a), f(b)\big) = m\Delta_x(a,b) \quad \Rightarrow \quad f(b) - f(a) = m\cdot(b-a) \quad \Rightarrow \quad f(b) = m\cdot(b-a) + f(a)$$

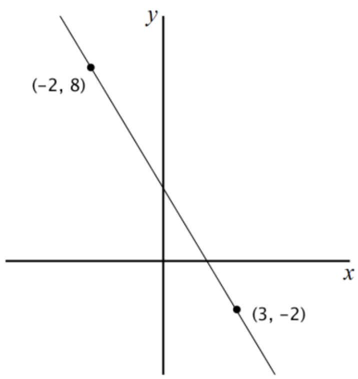
Consequently, for any input value $x = a$, we must have $f(x) = m\cdot(x-a) + f(a)$.

Problem 6. Suppose x and y represent measures of varying quantities in a process, and suppose that $y = f(x)$ is a linear function in the interval $-3 \le x \le 5$. If the constant rate of change in y with respect to x is $m = -\pi\sqrt{2}$ and we know $f(2.2) = \sqrt[3]{5}$, construct an algebraic formula for the function f.

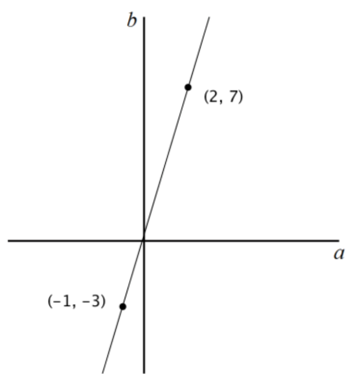
$$f(x) = -\pi\sqrt{2}\cdot(x-2.2) + \sqrt[3]{5}$$

Problem 7. For each trace below, construct an algebraic formula that gives the values of one variable as a function of the values of the other variable. Use proper function notation.

a.



b.



$(a)\ y = f(x) = -2(x-3) - 2$

$(b)\ \ b = f(a) = \dfrac{10}{3}(a+1) - 3$

[2] This formula is often called the *point-slope* representation for the linear function.