In the previous investigation, we explored the concept of "linear function." Of course, not all functions are linear; and, if a function is not linear, then there is no concept of "constant rate of change" that can apply to every pair of input values. Nonetheless, we often rely on linear thinking when considering such functions.

Problem 1. Sirius and Alexa ride their bicycles past a crosswalk, and Sirius passes Alexa when they are 3.5 feet from the crosswalk. Alexa passes Sirius 17 seconds later. Let d represent the measures of the quantity "the distance from the crosswalk, measured in feet" and let t represent the measures of the quantity "the time elapsed since Sirius passed Alexa, measured in seconds." Suppose that Alexa's distance-versus-time graph in the interval $0 \leq t \leq 18$ seconds is given by the following formula

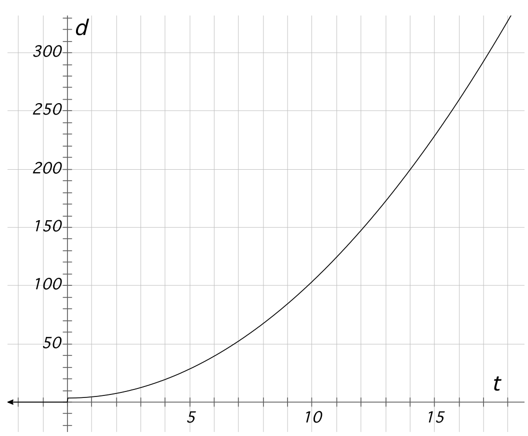
$$d = f(t) = 3.5 + t^2$$

Part (a). Fill in the table below.

$\Delta_t(a,b)$	Value of t	Value of $f(t)$	$\Delta_d(f(a), f(b))$
	0		
	0.75		
	2.25		
	3.00		
	4.50		
	5.25		

Does Alexa's distance from the crosswalk vary at a constant rate with respect to the number of seconds elapsed since Sirius passed her? Justify your response.

Part (b). The diagram below shows the graph of the function f on the input interval $0 \leq t \leq 18$ seconds. Let $d = g(t)$ be the function representing Sirius's distance-versus-time graph in the input interval $[0,18]$. If Sirius is riding at a *constant velocity*, draw the trace that represents the graph of the function g on this grid.



Part (c). What is Sirius's constant velocity? Explain the approach you used to answer this question.

Secant Line to the Graph of a Function

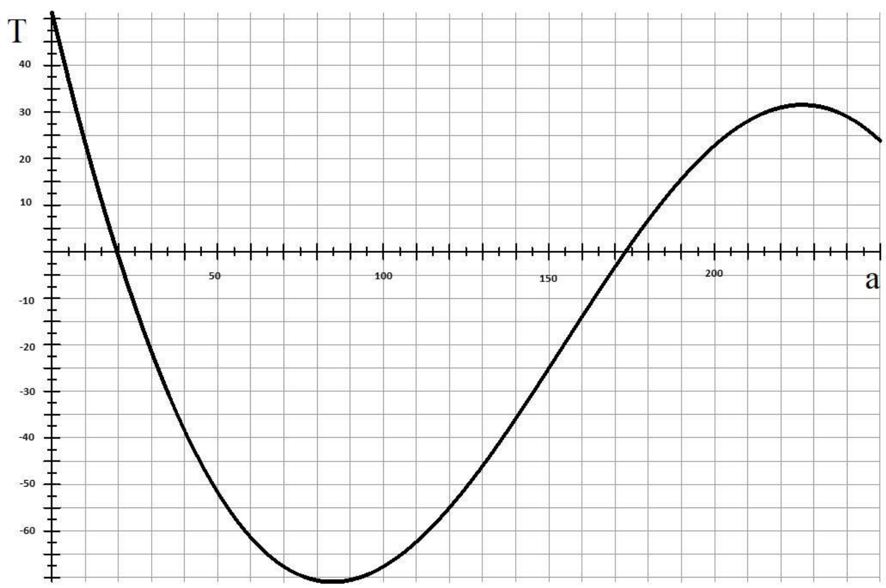
Suppose that $y = f(x)$ is a function in an input interval $a \leq x \leq b$, and suppose $f(a)$ and $f(b)$ are both defined. The *secant line* to the graph of the function f in the interval $[a, b]$ is the linear function $y = S(x)$ with the following property:

- As x varies from $x = a$ to $x = b$, the corresponding output of S varies from $f(a)$ to $f(b)$.

The word "secant" comes from the Latin verb "secare" which means "to cut."

The function $d = g(t)$ whose graph you drew in Problem 1 is the secant line to the graph of Alexa's distance versus time function in the interval $0 \leq t \leq 17$.

Minerva is piloting a commercial airliner from Nashville International Airport (BNA) to the Minneapolis International Airport. Let T represent measures of the quantity "the temperature of Minerva's aircraft, measured in degrees Fahrenheit," and let a represent measures of the quantity "the horizontal distance between Minerva's aircraft and BNA, measured in miles." The trace below shows the graph of the function $T = f(a)$ in the interval $0 \leq a \leq 250$.



Problem 2. Construct an algebraic formula for the secant line $T = S(a)$ to the graph of the function f in the interval $200 \leq a \leq 230$. What units are associated with the constant rate of change?

Average Rate of Change for a Function over an Interval

Suppose that $y = f(x)$ is a function in an input interval $a \leq x \leq b$, and suppose $f(a)$ and $f(b)$ are both defined. The *average rate of change* with respect to x for the outputs $f(x)$ over the interval $[a, b]$ is defined to be the constant rate of change in y with respect to x for the secant line $y = S(x)$ to the graph of f in this interval.

The constant rate of change for the outputs $f(x)$ with respect to x over the interval $[a, b]$ is given by the equation

$$m = \frac{\Delta_y\big(f(a), f(b)\big)}{\Delta_x(a, b)} = \frac{f(b) - f(a)}{b - a}$$

Problem 3. Suppose that the values of x and y are coordinated by the function $y = f(x) = x^2 - 3$ in the interval $-\infty < x < \infty$.

Part (a). Write an algebraic expression that represents the average rate of change $f(x)$ with respect to x over the input interval $a \leq x \leq b$. Use algebra to simplify your formula as much as possible.

Part (b). What is the average rate of change with respect to x for $f(x)$ over the interval $[-0.5, 1.5]$?

Problem 4. Justo is trying to give Mr. Hipsickles a fish oil supplement, and a drop of the fish oil lands in his water bowl. The fish oil spreads across the surface of the water as a circular disk. Let A represent the values of the quantity "the area measured in square millimeters for the disc." Let t represent the values of the quantity "the time passed since the disk began to spread, measured in seconds." Suppose we know $A = f(t)$ in the interval $0 \leq t \leq 3$ seconds, where the formula for f is given by

$$A = f(t) = \pi(7.84t)^{2/3}$$

Part (a). Consider the expression $\dfrac{f(t+3) - f(t)}{(t+3) - t}$.

Evaluate this expression for $t = 0.5$ and for $t = 2.35$ seconds. Describe the meaning of these values in the context of this problem.

Part (b). If h is nonzero real number, then what is the meaning of the following expression in the context of this problem?

$$\frac{f(t+h)-f(t)}{(t+h)-t}$$

What is the algebraic formula for this expression? Simplify your formula as much as you can.

Average Rate of Change Function

Let $y = f(x)$ be a function in an interval I, and let h be a nonzero constant. The *average rate of change* function

$$r = \text{ARC}_h(x) = \frac{f(x+h)-f(x)}{(x+h)-x} = \frac{f(x+h)-f(x)}{h}$$

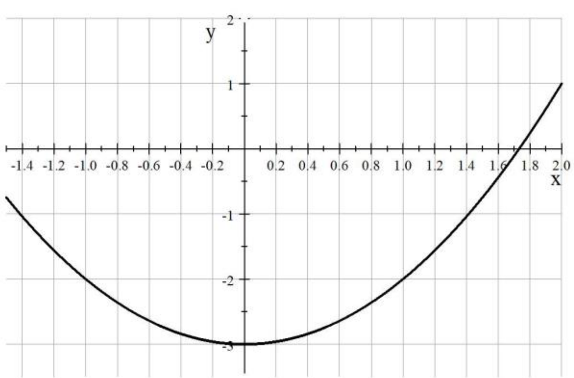
gives the average rate of change for $f(x)$ with respect to x over any subinterval from x to $x+h$.

Problem 5. Consider the function $y = f(x) = x^2 - 3$ in the interval $-\infty < x < \infty$.

Part (a). Construct the algebraic formula for the function $r = \text{ARC}_2(x)$. Use algebra to simplify your answer as much as possible.

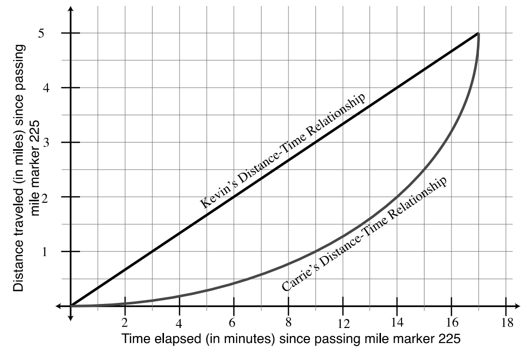
Part (b). What is the value of $\text{ARC}_2(-0.5)$? What is the meaning of this value?

Part (c). The trace below shows the graph of the function f in the interval $[-1.5,2]$. Represent $\text{ARC}_2(-0.5)$ on the this trace.



Homework.

Problem 1. The following trace represents the distance-time relationship for Kevin and Carrie as they cycled on a road from mile marker 225 to mile marker 230.



Part (a). How does the distance traveled and time elapsed compare for Carrie and Kevin as they traveled from mile marker 225 to mile marker 230?

Part (b). How do Carrie and Kevin's velocities compare as they travel from mile marker 225 to mile marker 230?

Part (c). How do Carrie and Kevin's average velocities compare over the time interval as they traveled from mile marker 225 to mile marker 230?

Problem 2. On a trip from Tucson to Phoenix via Interstate 10, you used your cruise control to travel at a constant velocity for the entire trip. Since your speedometer was broken, you decided to use your watch and the mile markers to determine your velocity.

- Let t represent the values of the quantity "the number of hours passed since you left Tucson."
- Let m represent the values of the quantity "the distance in miles between your car and Phoenix."

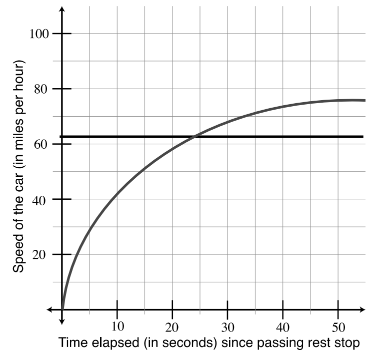
At mile marker 219 you noticed that the time on your digital watch just advanced to 9:22 AM. At mile marker 197 your digital watch had advanced to 9:46 AM.

Part (a). Compute the constant velocity at which you traveled over the time period from 9:22 AM to 9:46 AM. Why is your velocity *negative*?

Part (b). As you were passing mile marker 219 you also passed a truck. The same truck sped by you at exactly mile marker 197. Construct a distance-time graph of your car. On the same graph, construct one possible distance-time graph for the truck. Be sure to label the axes.

Part (c). Why are the average velocity of the car and the average velocity of the truck the same?

Problem 3. Blanche and Carlos are traveling to Nashville in separate cars. Carlos stops at a rest stop and waits for Blanche to pass. The graph that follows represents the velocities of their cars in terms of the elapsed time in seconds since Blanche passed the rest stop. Blanche's car is traveling at a constant velocity of 62 miles per hour. As she passes the rest stop, Carlos pulls out onto the highway; and they both continue traveling.



a. Which graph represents Blanche's velocity and which graph represents Carlos' velocity? Explain.

b. Which car is further down the road 20 seconds after being at the rest stop? Explain.

c. Explain the meaning of the intersection point.

d. What is the relationship between the positions of Blanche's car and Carlos' car 27 seconds after passing the rest stop?

e. Carlos catches up with Blanche 64.5 seconds after she passed the rest stop. What is the average velocity of Carlos' car over the interval from 0 seconds to 64.5 seconds after leaving the rest stop? Explain.

Problem 4. Let d represent measures of the quantity "the distance of LaRue's Ford F150 (measured in feet) from mile marker 420 on FM Road 18," and let t represent measures of the quantity "the time elapsed (in seconds) since LaRue's Ford F150 passed mile marker 420." Suppose that $d = f(t)$ in the interval $[0,120]$. The algebraic formulas below represent various ways the function could coordinate the values of d and t. For each of the formulas, determine the average velocity of the car over the specified time interval.

Part (a). $d = f(t) = t^2$ over the time interval from $t = 5$ to $t = 30$ seconds

Part (b). $d = f(t) = 4.17 \cdot 2.8^t$ over the time interval from $t = 0$ to $t = 3$ seconds

Part (c). $d = f(t) = 3.75\cos(t)$ over the time interval from $t = 10$ to $t = 40$ seconds

Part (d). $d = f(t) = 2(1+t)^{-1}$ over the time interval from $t = 1$ to $t = 1 + h$ seconds

Problem 5. Let P represent the values of the quantity "the population of Telluride Colorado measured in thousands"; and let t represent the values of the quantity "the time passed since January 1, 1990 measured in years." Suppose that, in the interval $0 \le t \le 10$, we know that the values of P are related to the values of t according to the function

$$P = f(t) = 1.645 \cdot (1.06)^t$$

Part (a). Construct the formula for the function $r = \text{ARC}_1(t)$ that determines the average rate of change of $f(t)$ with respect to t over time intervals from t to $t + 1$ years.

Part (b). Construct the formula for the function $r = \text{ARC}_{-0.203}(t)$ that determines the average rate of change of $f(t)$ with respect to t over time intervals from t to $t - 0.203$ years.

Part (c). Use your graphing calculators to sketch traces of the graphs of the functions ARC_1 and $\text{ARC}_{-0.203}$ in the interval $0.25 \le t \le 9$ years. Why can't we use the interval $[0,10]$ for both graphs?

Entering an Average Rate of Change Function into Your TI83 or TI84 Calculator

1. Call up the function menu, and enter the formula for f into an available function. For example,

$$Y_1 = 1.645 * 1.06\text{^}x$$

2. Select another available function and enter the average rate of change formula, referencing the function Y_1. For example,

$$Y_2 = \big(Y_1(x+h) - Y_1(x)\big)/h$$

To reference the function Y_1 in the formula for Y_2, find the VARS button on your calculator and follow the sequence

VARS → YVARS → FUNCTION → Y_1

Problem 6. Consider the functions $y = f(x)$ in the interval $-\infty < x < \infty$ whose formulas are given below. For each function, construct the algebraic formula for $r = \text{ARC}_h(x)$ that gives the average rate of change for the function $f(x)$ with respect to x on the input interval from x to $x + h$. Use algebra to simplify your formula as much as possible.

Part (a). $y = f(x) = 12x + 6.5$

Part (b). $y = f(x) = 97$

Part (c). $y = f(x) = 3x^2 - 9$

Part (d). $y = f(x) = 2x^{-1}$

Answers to the Homework.

Problem 1.

Part (a). Both Carrie and Kevin traveled the same distance (5 miles) and they took the same amount of time (17 minutes) to travel those 5 miles.

Part (b). Kevin's velocity is constant for the entire 17-minute period, Carrie traveled slower in the beginning and gradually sped up.

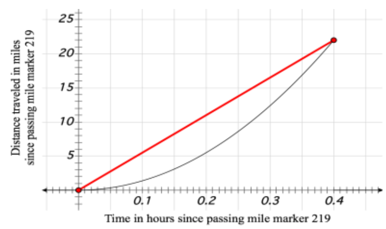
Part (c). Their average velocities were the same (5/17 miles per minute).

Problem 2.
Part (a). The constant velocity would be

$$\frac{197-219}{24/60} = -55 \text{ mph}$$

The velocity value is negative because you are approaching Phoenix; therefore the values of m are decreasing as the values of t increase.

Part (b). Here is the graph. Your distance versus time graph for the truck could vary.



Part (c). The average velocity measures the constant rate at which an object would need to travel in order to produce the same net change in distance in an identical segment of time. Since we only measured the position of the truck and car at the ends of the segment, and they were next to each other at those points and both had traveled the same amount of time, the average velocities are the same.

Problem 3.

Part (a). Since this is a velocity versus time graph, a constant velocity of 65 mph is represented by a horizontal line at $y = 6.5$. So the horizontal line represents Blanche's velocity. Carlos' velocity increases from 0 mph at $t = 0$, which is represented by the non-horizontal graph. This graph begins at the origin and the velocity of the car increases as the number of seconds since passing the rest stop increases.

Part (b). Twenty seconds after passing the rest stop, Blanche is further down the road because during the entire 20-second interval Blanche was traveling at a velocity greater than that of Carlos.

Part (c). The intersection point represents the number of seconds that have elapsed since passing the rest stop when the two cars are traveling the same velocity, namely 62 mph.

Part (d). Twenty-seven seconds after being at the rest stop, the velocity of Carlos is just slightly greater than the velocity of Blanche. However, this does not imply that Carlos is ahead of Blanche.

Part (e). Since Blanche and Carlos are in the same position 64.5 seconds after both cars past the rest stop, the average velocity of Carlos over this interval is the constant velocity of Blanche, namely 62 miles per hour.

Problem 4.

Part (a). The average velocity will be
$$\frac{\Delta_d\big(f(5),f(30)\big)}{\Delta_t(5{,}30)}=\frac{f(30)-f(5)}{30-5}=35\text{ feet per second}$$

Part (b). The average velocity will be
$$\frac{\Delta_d\big(f(0),f(3)\big)}{\Delta_t(0{,}3)}=\frac{f(3)-f(0)}{3-0}\approx 29.12\text{ feet per second}$$

Part (c). The average velocity will be
$$\frac{\Delta_d\big(f(10),f(40)\big)}{\Delta_t(10{,}40)}=\frac{f(40)-f(10)}{40-10}\approx 0.0215\text{ feet per second}$$

Part (d). The average velocity will be
$$\frac{\Delta_d\big(f(1),f(1+h)\big)}{\Delta_t(1{,}1+h)}=\frac{f(1+h)-f(1)}{(1+h)-1}=\frac{\left({}^{2}\!/_{(1+h)}\right)-2}{h}=-\frac{2}{1+h}\text{ feet per second}$$

Problem 5. We have
$$r=\mathrm{ARC}_1(t)=\frac{f(t+1)-f(t)}{1}=1.645(1.06^{t+1}-1.06^{t})$$
$$r=\mathrm{ARC}_{-0.203}(t)=\frac{f(t-0.203)-f(t)}{-0.203}=-8.103(1.06^{t-0.203}-1.06^{t})$$

We have to restrict the interval on which the average rate of change functions are defined because the fixed values of h would otherwise take us outside the interval on which the formula for f is valid. (We could use the interval $0\le t\le 9$ for the first function, and $0.203\le t\le 10$ for the second.)

Problem 6.

Part (a). We know that
$$r=\mathrm{ARC}_h(x)=\frac{f(x+h)-f(x)}{h}=\frac{[12(x+h)+6.5]-[12x+6.5]}{h}=12\quad(h\neq 0)$$

Part(b). We know that
$$r=\mathrm{ARC}_h(x)=\frac{f(x+h)-f(x)}{h}=\frac{[97]-[97]}{h}=0\quad(h\neq 0)$$

Part (c). We know that
$$r=\mathrm{ARC}_h(x)=\frac{f(x+h)-f(x)}{h}$$

$$= \frac{[3(x+h)^2 - 9] - [3x^2 - 9]}{h}$$

$$= \frac{[3(x^2 + 2xh + h^2) - 9] - [3x^2 - 9]}{h} = 9x^2 + 6xh + 3h^2 \qquad (h \neq 0)$$

Part (d). We know that

$$r = \mathrm{ARC}_h(x) = \frac{f(x+h) - f(x)}{h}$$

$$= \left(\frac{1}{h}\right)\left(\frac{2}{x+h} - \frac{2}{x}\right)$$

$$= \left(\frac{1}{h}\right)\left(\frac{2x - 2(x+h)}{x(x+h)}\right) = -\frac{2}{x(x+h)} \qquad (h \neq 0)$$