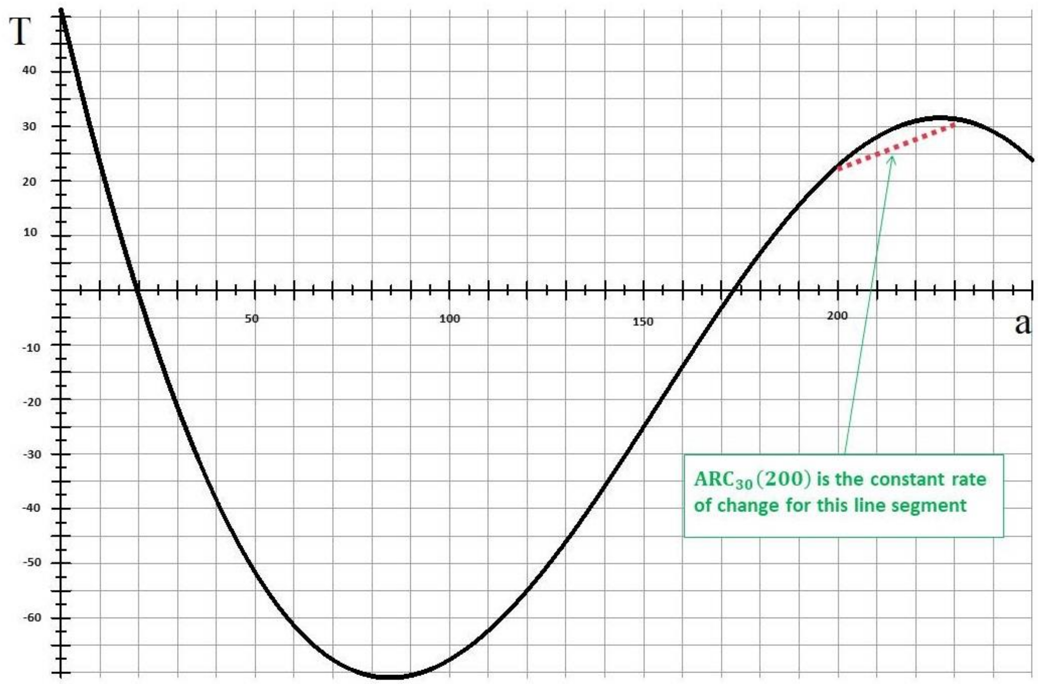
In the sciences, average rate of change is very important, and the primary reason for this is simple. Suppose that x and y represent the values of two varying quantities in a physical process, and suppose that $y = f(x)$ in an interval. In practice, it is usually easier to measure (and thus construct) average rate of change functions $r = \text{ARC}_h(x)$ instead of gathering data to construct the actual function f. The question for scientists is then "what information can we learn about the function f by examining the average rate of change functions?

Problem 1. Let's think again about Minerva's flight to Minneapolis in the last investigation. Recall that we let T represent measures of the quantity "the temperature of Minerva's aircraft, measured in degrees Fahrenheit," and let a represent measures of the quantity "the horizontal distance between Minerva's aircraft and BNA, measured in miles."



Part (a). The function $\boldsymbol{r = \text{ARC}_{30}(a)}$ gives the average rate of change for $\boldsymbol{f}$ with respect to $\boldsymbol{a}$ over the interval $[\boldsymbol{a}, \boldsymbol{a + 30}]$. Use the trace above to fill in the following tables.

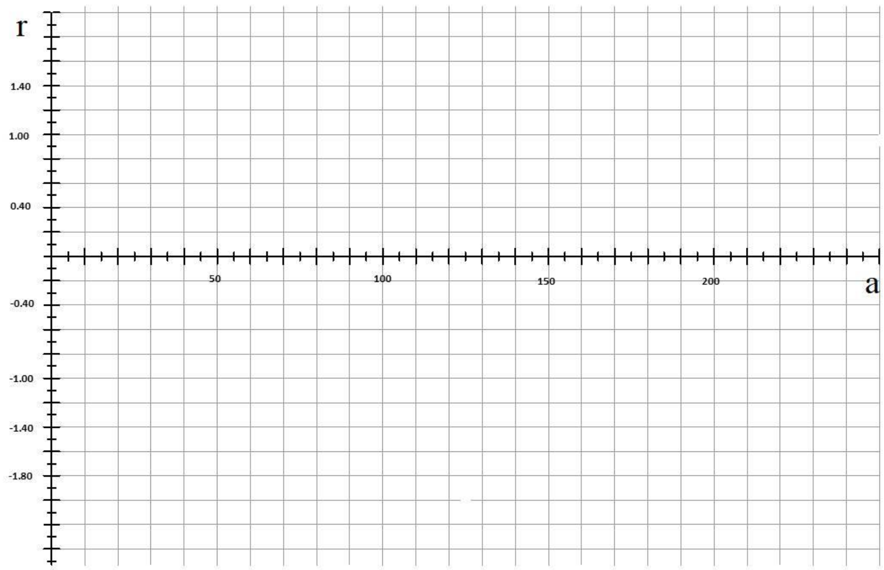
a	$\text{ARC}_{30}(a)$
0	
30	
60	
90	
120	

a	$\text{ARC}_{30}(a)$
50	
80	
110	
140	
170	

a	$\text{ARC}_{30}(a)$
100	
130	
160	
190	
220	

Part (b). What are the units associated with the output $\mathbf{ARC_{30}}(\boldsymbol{a})$?

Part (c). Plot the data in the tables on the grid below and use these points to sketch a possible trace for the graph of the function $\boldsymbol{r} = \mathbf{ARC_{30}}(\boldsymbol{a})$.

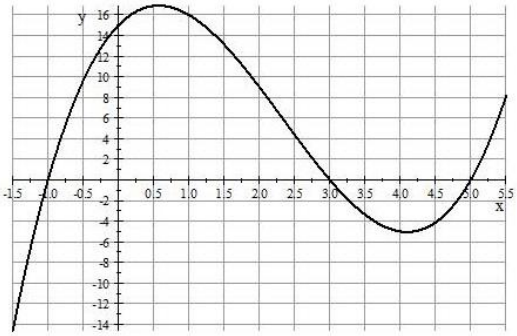


Part (d). Use your trace to predict the output value $\mathbf{ARC_{30}}(\mathbf{70})$.

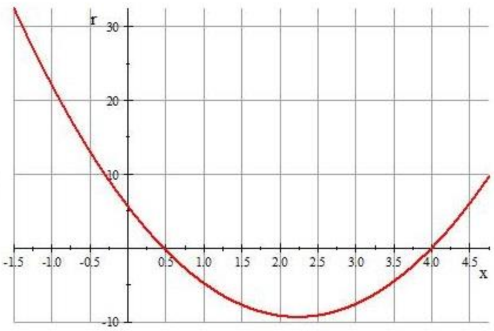
Part (e). Use the graph $\boldsymbol{T} = \boldsymbol{f}(\boldsymbol{a})$ to determine the actual output value $\mathbf{ARC_{30}}(\mathbf{70})$.

Problem 2. Consider your trace of the graph for $r = \mathrm{ARC}_{30}(a)$. There are two input values where the output of ARC_{30} changes from positive to negative. What are some things you observe about the outputs of the function f in the vicinity of these input values?

Problem 3. The diagram on the left shows the graph of the function $y = f(x) = x^3 - 7x^2 + 7x + 15$, and the diagram on the right shows the average rate of change function $r = \text{ARC}_{0.25}(x)$ for the function f.



Graph of f



Graph of $\text{ARC}_{0.25}$

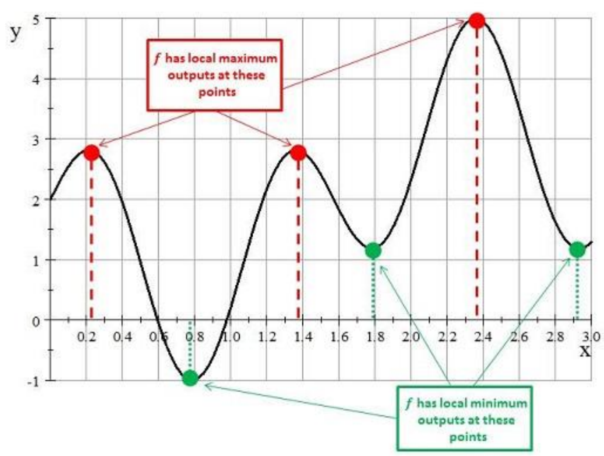
Once again, there are two input values where the output of $\text{ARC}_{0.25}$ changes from positive to negative. Is the behavior of the function f in the vicinity of these inputs consistent with what you observed in Problem 2? Explain.

Local Extrema for a Function

Let $y = f(x)$ be a function.

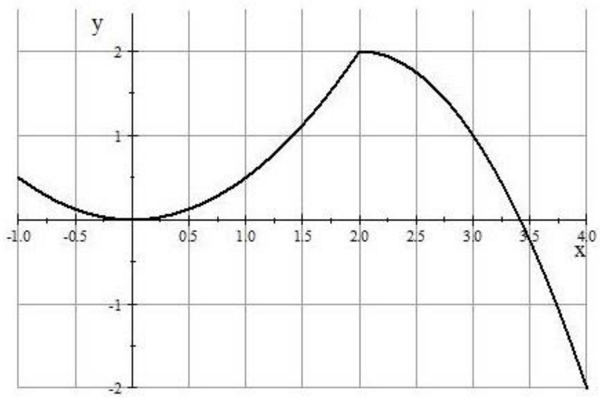
- We say that the function f has a *local maximum output* at an input value $x = a$ provided $f(a)$ is the largest output for f on some small input interval containing $x = a$.
- We say that the function f has a *local minimum output* at an input value $x = a$ provided $f(a)$ is the smallest output for f on some small input interval containing $x = a$.

We often say that the function f has a *local extremum* at an input value $x = a$ when we know that f has a local maximum or minimum output at $x = a$ but have not yet determined which one it is.

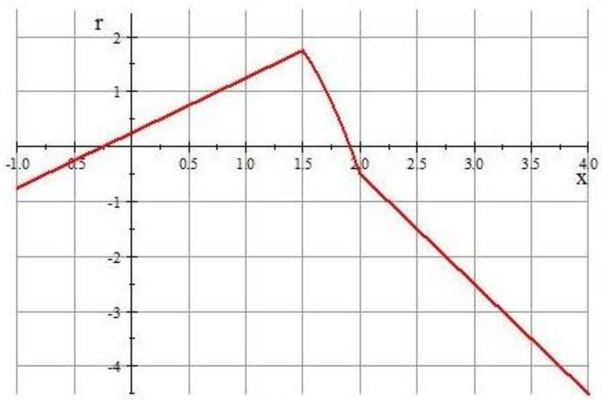


Problem 4. Consider the function f in Problem 3 along with the average rate of change function given. How could you use the average rate of change function to predict the approximate input value x where the function f has a *local* maximum or minimum output? What in your strategy allows you to tell the difference?

Problem 5. The diagram below shows the graph of a function $y = f(x)$ along with its average rate of change function $r = \text{ARC}_{0.50}(x)$.



Graph of f



Graph of $\text{ARC}_{0.50}$

Does the strategy you devised in Problem 4 also work in this case? If not, how would you make changes in your strategy so that it works in both problems?

A manufacturing plant is dumping pollutants into a large lake. As part of an EPA settlement, the plant must lower the mass of pollutant in the lake by doing two things.

- The plant must reduce the mass of pollutants it dumps into the lake.
- The plant must install a filtration system to remove existing pollutants from the lake.

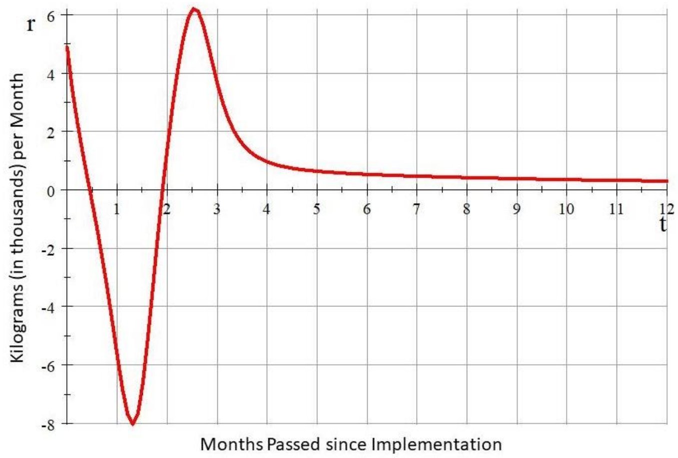
Scientists must monitor the total mass of pollutant in the lake as a function of the time passed since the reduction measures were put in place.

The scientists can directly measure the mass of pollutant entering the lake, and they can directly measure the mass of pollutant that is being filtered from the lake. They set up monitoring systems to measure these quantities every day.

Pollutant can enter the lake only from the plant's dumping process, and it can leave the lake only through the filtration system. Therefore, the difference in the measurements of the dumping and filtration monitors represents the average rate of change in the mass of pollutant in the lake per day.

Let t represent the values of the quantity "the number of months passed since the reduction measures were implemented." Let m represent the values of the quantity "the mass in thousands of kilograms of pollutant in the lake", and let $m = f(t)$ represent m as a function of t.

Finally, let $r = \text{ARC}_{0.0333}(t)$ denote the average rate of change function created from the data values. (One day is approximately 0.0333 of one month.) The diagram below shows a trace of the graph of the function $\text{ARC}_{0.0333}$.



Problem 6. The point $(2,1.50)$ lies on the graph above. Does this mean that there are 1,500 kilograms of pollutant in the lake two months after implementation? Explain.

Problem 7. Based on this graph, at what approximate input value t did the function f have a local minimum output? How did you decide?

Problem 8. Based on this graph, would it be correct to say that the mass of pollutant in the lake was decreasing on the input interval $2.5 < t \leq 12$? Explain.

Problem 9. On which approximate input intervals is the output of the function f increasing? What does this behavior mean in the context of this problem?

Problem 10. The EPA claims that the plant is not living up to the terms of the settlement. Is the EPA justified in making this claim? Explain.

Homework.

Problem 1. Consider the function $y = f(x) = 2x - x^2$. Construct the formula for the average rate of change function

$$r = \mathrm{ARC}_{0.1}(x) = \frac{f(x+0.1) - f(x)}{0.1}$$

Use algebra to simplify this formula as much as possible.

Problem 2. Consider the function $m = f(n) = 3 - 2n - n^2$. Construct the formula for the average rate of change function

$$r = \mathrm{ARC}_{-2}(n) = \frac{f(n-2) - f(n)}{-2}$$

Use algebra to simplify this formula as much as possible.

Problem 3. Consider the function $a = f(b) = b^{-1}$. Construct the formula for the average rate of change function $r = \mathrm{ARC}_h(b)$, where h is a nonzero constant. Use algebra to simplify this formula as much as possible.

Problem 4. Consider a function $y = f(x)$. Suppose we know that $f(1) = 2.8$, and suppose we know that $\mathrm{ARC}_{0.5}(1) = -1.6$.

Part (a). What is the linear formula for the secant line to the graph of f that passes through the points $\big(1, f(1)\big)$ and $\big(1.5, f(1.5)\big)$? [The *point-slope formula* from Investigation 3 is helpful here. Look it up if you don't remember it.]

Part (b). What is the value of $f(1.5)$?

Problem 5. Consider a function $y = f(x)$. Suppose we know that $\mathrm{ARC}_{1.2}(x) = x^2 - 1$.

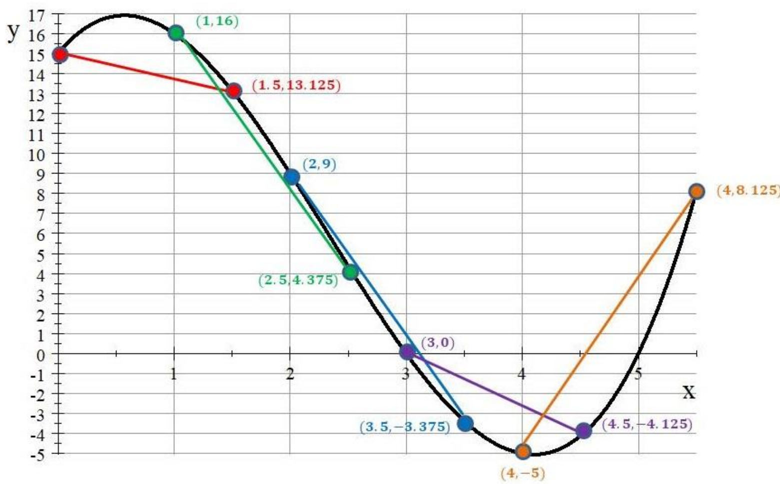
Part (a). If we know that $f(2) = -3.74$, what is the linear formula for the secant line to the graph of f that passes through the points $\big(2, f(2)\big)$ and $\big(3.2, f(3.2)\big)$?

Part (b). What is the value of $f(3.2)$?

Part (c). Based on your answer to Part (b), what is the linear formula for the secant line to the graph of f that passes through the points $\big(3.2, f(3.2)\big)$ and $\big(4.4, f(4.4)\big)$?

Part (d). What is the value of $f(4.4)$?

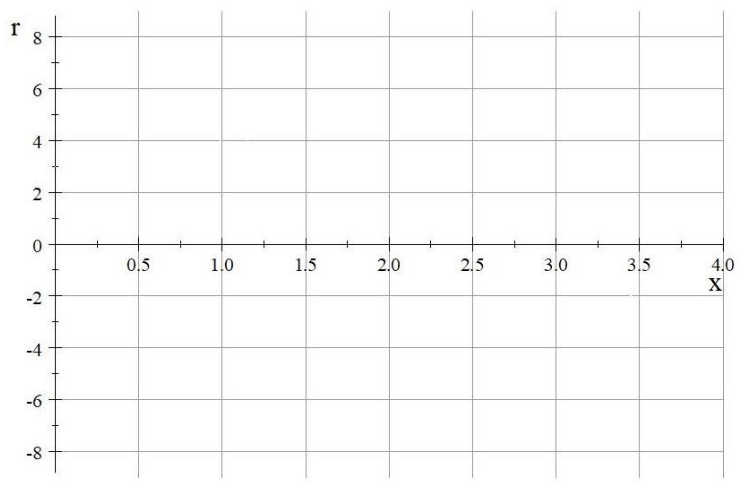
Problem 6. The diagram below shows the graph of a function $y = f(x)$ along with secant lines to the graph of f on input intervals from x to $x + 0.5$ for several values of x.



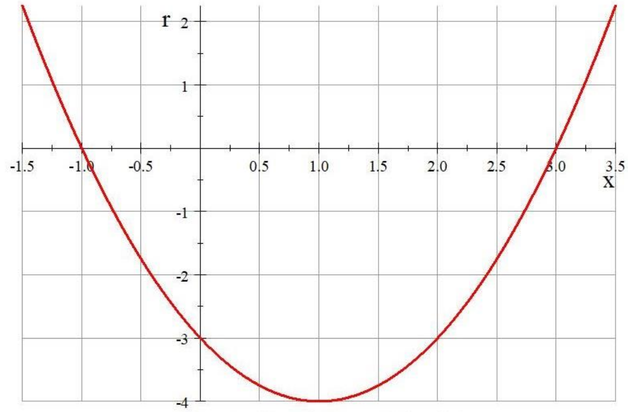
Value of x	Value of $f(x)$	Value of $f(x+1.5)$	Value of $\text{ARC}_{1.5}(x)$
0			
1			
2			
3			
4			

Part (a). Use the information from the graph to fill in the table above.

Part (b). Use the information in your table to construct sketch of the function $r = \text{ARC}_{1.5}(x)$ on the grid provided.



Problem 7. The graph below shows an average rate of change function for a function $y = f(x)$.



Graph of Function $r = \text{ARC}_{0.75}(x)$

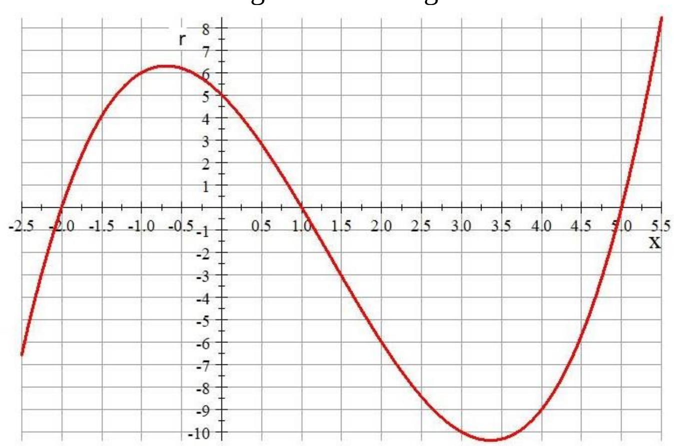
Part (a). What is the meaning of the point $(2.0, -3)$ above?

Part (b). If we know that $f(2.0) = 6.2$, what is the value of $f(2.75)$?

Part (c). What is the approximate input value x where the function f has a local maximum output?

Part (d). What is the approximate input value x where the function f has a local minimum output?

Problem 8. The graph below shows an average rate of change function for a function $y = f(x)$.



Graph of Function $r = \text{ARC}_{0.33}(x)$

Part (a). What is the meaning of the point $(-1.0, 6.0)$ above?

Part (b). If we know that $f(2.0) = 1.8$, what is the value of $f(2.33)$?

Part (c). On what approximate input intervals is the output of the function f increasing?

Part (d). On what approximate input intervals is the output of the function f decreasing?

Answers to the Homework.

Problem 1. Based on the definition of the average rate of change functions, we know

$$\begin{aligned}\mathrm{ARC}_{0.1}(x) &= \frac{f(x+0.1)-f(x)}{0.1}\\ &= \frac{[2(x+0.1)-(x+0.1)^2]-[2x-x^2]}{0.1}\\ &= \frac{2x+0.2-x^2-0.2x-0.01-2x+x^2}{0.1}\\ &= \frac{-0.2x+0.19}{0.1}\\ &= -2x+1.9\end{aligned}$$

Problem 2. Based on the definition of the average rate of change functions, we know

$$\begin{aligned}\mathrm{ARC}_{-2}(n) &= \frac{f(n-2)-f(n)}{-2}\\ &= \frac{[3-2(n-2)-(n-2)^2]-[3-2n-n^2]}{-2}\\ &= \frac{3-2n+4-n^2+4n-4-3+2n+n^2}{-2}\\ &= \frac{4+4n-4}{-2}\\ &= -2n\end{aligned}$$

Problem 3. Based on the definition of the average rate of change functions, we know

$$\begin{aligned}\mathrm{ARC}_{h}(b) &= \frac{f(b+h)-f(b)}{h}\\ &= \left(\frac{1}{h}\right)\left(\frac{1}{b+h}-\frac{1}{b}\right)\\ &= \left(\frac{1}{h}\right)\left(\frac{b-(b+h)}{b(b+h)}\right)\\ &= \left(\frac{1}{h}\right)\left(\frac{-h}{b(b+h)}\right)\\ &= -\frac{1}{b(b+h)}\end{aligned}$$

Problem 4.

Part (a). By definition, $\text{ARC}_{0.5}(1)$ represents the average rate of change for the function f on the input interval from $x = 1$ to $x = 1.5$. Therefore, $\text{ARC}_{0.5}(1)$ is also the constant rate of change for the secant line that passes through the points $(1, f(1))$ and $(1.5, f(1.5))$. Since we have the slope of this line along with one point on the line (namely (1,2.8)) we can construct the point-slope formula for the secant line.

$$y - 2.8 = -1.6(x - 1)$$

Solving this equation for y gives us $y = -1.6(x - 1) + 2.8$. (You can simplify this formula further to obtain the *slope-intercept formula* if you wish, but this is not necessary.)

Part (b). Since we have the formula for the secant line that passes through the points $(1, f(1))$ and $(1.5, f(1.5))$, we know that

$$f(1.5) = -1.6(1.5 - 1) + 2.8 = 2$$

Problem 5.

Part (a). We approach this problem in the same way we approached Part (a) of Problem 4. The slope of the secant line is

$$\text{ARC}_{1.2}(2) = 2^2 - 1 = 3$$

Therefore, the linear formula for the secant line is $y = 3(x - 2) - 3.74$.

Part (b). We know that $f(3.2) = 3(3.2 - 2) - 3.74 = -0.14$.

Part (c). Since $4.4 = 3.2 + 1.2$, we know that the slope of the secant line passing through the points $(3.2, f(3.2))$ and $(4.4, f(4.4))$ is given by $\text{ARC}_{1.2}(3.2)$. Now, $\text{ARC}_{1.2}(3.2) = 9.24$. Thus, the linear formula for the secant line is $y = 9.24(x - 3.2) - 0.14$.

Part (d). We know that $f(4.4) = 9.24(4.4 - 3.2) - 0.14 = 10.948$.

Problem 6.

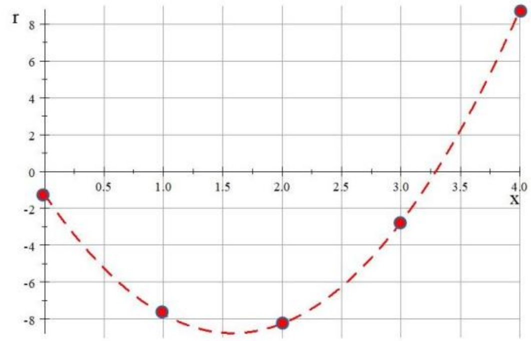
Part (a). The value of $\text{ARC}_{1.5}(x)$ is the constant rate of change for the secant line that passes through the points $(x, f(x))$ and $(x + 1.5, f(x + 1.5))$. For example,

$$\text{ARC}_{1.5}(2) = \frac{f(3.5) - f(2)}{1.5} = -8.25$$

With this in mind, we can fill in the table.

Value of x	Value of $f(x)$	Value of $f(x + 1.5)$	Value of $\text{ARC}_{1.5}(x)$
0	15	13.125	-1.25
1	16	4.375	-7.75
2	9	-3.375	-8.25
3	0	-4.125	-2.75
4	-5	8.125	8.75

Part (b). Here is a sketch of the function $\text{ARC}_{1.5}$ based on the data in the table from Part (a).



Problem 7.
Part (a). We know that $\text{ARC}_{0.75}(2.0) = -3$. This tells us that the average rate of change for the function f on the input interval from $x = 2$ to $x = 2.75$ is $r = -3$.

Part (b). We know $f(2.75) = -3(2.75 - 2) + 6.2 = 3.95$.

Part (c). The function f will have a local maximum output for an input value that is *close* to any input value $x = a$ where the output of the function $\text{ARC}_{0.75}$ changes from positive to negative. Based on the graph of the function $\text{ARC}_{0.75}$ shown above, we may conclude that the function f has a local maximum output when $x \approx -1.0$. (We cannot say for certain what the exact input value will be.)

Part (d). The function f will have a local minimum output for an input value that is *close* to any input value $x = a$ where the output of the function $\text{ARC}_{0.75}$ changes from negative to positive. Based on the graph of the function $\text{ARC}_{0.75}$ shown above, we may conclude that the function f has a local minimum output when $x \approx 3.0$. (We cannot say for certain what the exact input value will be.)

Problem 8.
Part (a). We know that $\text{ARC}_{0.33}(-1.0) = 6.0$. This tells us that the average rate of change for the function f on the input interval from $x = -1$ to $x = -0.67$ is $r = 6.0$.

Part (b). We see from the graph provided that $\text{ARC}(2.0) = -6$; therefore, we know $f(2.33) = -6(2.33 - 2) + 1.8 = -0.18$.

Part (c). The graph of the function f will be increasing on the approximate input intervals where the output of the function $\text{ARC}_{0.33}$ is positive. Thus, we know that the function f is increasing on the approximate input intervals $-2 < x < 1$ and $5 < x < +\infty$. (It is the endpoints of the intervals that are approximate).

Part (d). The graph of the function f will be decreasing on the approximate input intervals where the output of the function $\text{ARC}_{0.33}$ is negative. Thus, we know that the function f is decreasing on the approximate input intervals $-\infty < x < -2$ and $1 < x < 5$. (It is the endpoints of the intervals that are approximate).