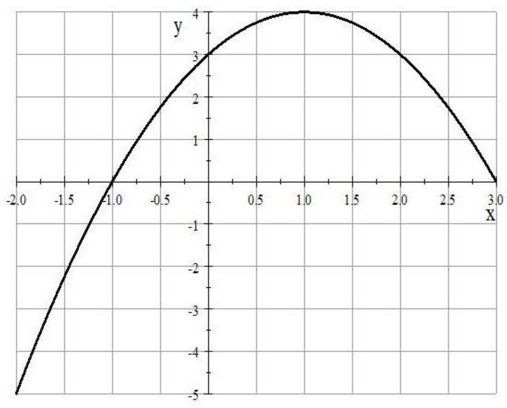
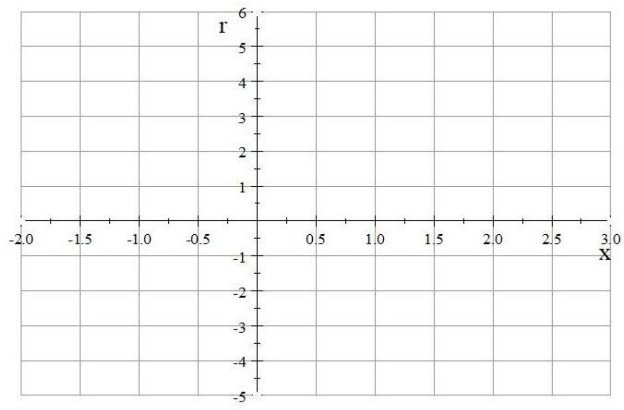
In the previous investigation, we saw that average rate of change functions for a function $y = f(x)$ carry useful information about the function. If we have an average rate of change function $r = \text{ARC}_h(x)$ then the input intervals on which ARC_h has positive or negative output correspond roughly to intervals where the function f is increasing or decreasing output. This information can be helpful in drawing conclusions about the quantities related by the function f; unfortunately, this information is only approximate. Fortunately, there is a way around this problem.

Problem 1. The trace on the left below shows the graph of the function $y = f(x) = 4 - (x-1)^2$ in the interval $-2 \le x \le 3$.



Graph of *f*



Graphs of average rate of change functions

Part (a). On your graphing calculator, use the following steps to construct average rate of change functions for the function f.

- On the home screen, type the following list exactly as it appears, then perform the indicated actions.
  {2.0,1.5,1.0,0.5,0.25,0.1,0.01} <Press STO> <Press L_1>
  (This action creates a *list variable* on your calculator.)
- Go to the function screen; select Y_1 and enter the formula for f. Turn this function OFF.
- Select Y_2 and enter the following formula:
  $$(Y_1(X + L_1) - Y_1(X)) \;/\; L_1$$
  To call up Y_1 --- <Press Vars> Select Y-Vars, then select Y_1.
  To call up L_1 --- <Press 2nd> <Press 1>
- Set the viewing window so that you have $-2 \le X \le 3$ Xscl = 1 $-5 \le Y \le 6$ YScl = 1
- <Press Graph>
  You should see a sequence of traces appearing on the screen.

Part (b). The first trace to appear is the graph of $r = \text{ARC}_1(x)$; the last is the graph of $r = \text{ARC}_{0.01}(x)$. Carefully sketch each of these traces on the grid provided above.

Part (c). Each of the functions in this sequence can be used to predict approximately which input value produces the local maximum output for f. What can you say about the predictive power of these functions?

Part (d). Now, have your graphing calculator sketch the graph of $y = Y_3(x) = 2 - 2x$ along with the sequence. For any given input value, what can you say about the how the outputs of these functions relate?

Problem 2. Re-enter your list so that $L_1 = \{0.5, 0.25, 0.10, 0.01, 0.001\}$ and reset your function to $Y_1 = \sin(x)$. Reset your viewing window so that you have $0 \le X \le 2\pi$ XScl $= \pi/2$ and $-1 \le Y \le 1$ YScl $= 1$. (Make sure your calculator is in radian mode.)

Part (a). Sketch the graph of the function Y_1. On what input intervals is the output of this function increasing? On what input intervals is the output decreasing?

Part (b). Now, turn off the function Y_1 and sketch the sequence of five average rate of change functions determined by L_1. Does the predictive power of these functions improve as the value of h gets smaller? Explain.

Part (c). Let $Y_3 = \cos(x)$ and have the calculator graph the average rate of change functions along with this function. What do you notice?

The Derivative Function for a Function

Let $y = f(x)$ be a function in an interval I. The *derivative function* for the function f in the interval I is defined by the limiting process

$$r = f'(x) = \lim_{h \to 0} \mathrm{ARC}_h(x)$$

When this limiting process produces a numeric value for an input value x, we say that the function f is *differentiable* at the input value x. The derivative function for f is also commonly denoted by the symbol

$$r = \frac{df}{dx}(x)$$

When the derivative function $r = f'(x)$ for a function $y = f(x)$ is defined for an input value $x = a$, it is customary to call the output $f'(a)$ the *instantaneous rate of change* of the output for f with respect to x at the input value $x = a$.

In Problem 2, we considered the function $y = f(x) = \sin(x)$, and we observed that the outputs of average rate of change functions for this function get closer and closer to the corresponding outputs of the function $y = g(x) = \cos(x)$ as the values of h get closer and closer to 0. We therefore consider the function g to be the derivative function for $f(x) = \sin(x)$.[1] It is customary to write

$$\frac{df}{dx}(x) = \lim_{h \to 0} \mathrm{ARC}_h(x) = \lim_{h \to 0} \frac{\sin(x+h) - \sin(x)}{h} = \cos(x)$$

[1] We will leave a careful proof of this fact to a course on the theory of calculus.

Problem 3. On your graphing calculator, let $Y_1 = \cos(x)$. Keep your viewing window and L_1 the same as in Problem 2.

Part (a). Sketch the graph of the function Y_1. On what input intervals is the output of this function increasing? On what input intervals is the output decreasing?

Part (b). Now, turn off the function Y_1 and sketch the sequence of five average rate of change functions determined by L_1. Does the predictive power of these functions improve as the value of h gets smaller? Explain.

Part (c). Let $Y_3 = \sin(x)$ and have the calculator graph the average rate of change functions along with this function. What do you notice?

Part (d). If we let $y = f(x) = \cos(x)$ in the interval $-\infty < x < \infty$, what do you think the formula for $r = f'(x)$ should be? Explain your thinking.

Problem 5. On your graphing calculator, let $Y_1 = e^x$. (This is the base-*e* *exponential* function.) Reset your viewing window to $-3 \le X \le 3$ XScl = 0.5 and $0 \le Y \le 10$ YScl = 1. Keep L_1 the same as in Problem 2.

Part (a). Turn off the function Y_1 and sketch the sequence of five average rate of change functions determined by L_1.

Part (b). Based on these graphs, what function do you think $\lim_{h \to 0} \mathrm{ARC}_h(x)$ should produce? Justify you answer.

Problem 6. On your graphing calculator, let $Y_1 = \ln(x)$. (This is the base-*e* logarithm.) Reset your viewing window to $0 \le X \le 4$ XScl = 0.5 and $0 \le Y \le 5$ YScl = 1. Keep L_1 the same as in Problem 2.

Part (a). Turn off the function Y_1 and sketch the sequence of five average rate of change functions determined by L_1.

Part (b). Based on these graphs, what function do you think $\lim_{h\to 0} \text{ARC}_h(x)$ should produce? Justify you answer.

Part (c). Turn off the average rate of change functions and sketch the graph of $Y_1 = \ln(x)$. Is there any input interval where the output of Y_1 is decreasing? Does your proposed derivative formula support this? Explain.

Problem 7. Let $y = f(x)$ be a function in an interval I, and consider its derivative function $r = f'(x)$. Complete the following sentences and justify your answer based on average rate of change functions.

- If the output of the function f is increasing in a subinterval of I, then the output of f' is
- If the output of the function f is decreasing in a subinterval of I, then the output of f' is
- If the function f has a local maximum output at $x = a$, then the output of f'
- If the function f has a local minimum output at $x = a$, then the output of f'

Problem 8. You and a friend are driving down the highway. Your friend asks you how fast you are driving, and you say "I am driving 68 miles per hour." Are you referring to instantaneous velocity or average velocity when you make this statement? Explain your thinking.

Some Specific Derivative Formulas

- If $y = f(x) = \sin(x)$ in the interval $-\infty < x < \infty$, then $\frac{df}{dx}(x) = \cos(x)$
- If $y = f(x) = \cos(x)$ in the interval $-\infty < x < \infty$, then $\frac{df}{dx}(x) = -\sin(x)$
- If $y = f(x) = e^x$ in the interval $-\infty < x < \infty$, then $\frac{df}{dx}(x) = e^x$
- If $y = f(x) = \ln(x)$ in the interval $0 < x < \infty$, then $\frac{df}{dx}(x) = x^{-1}$

Homework.

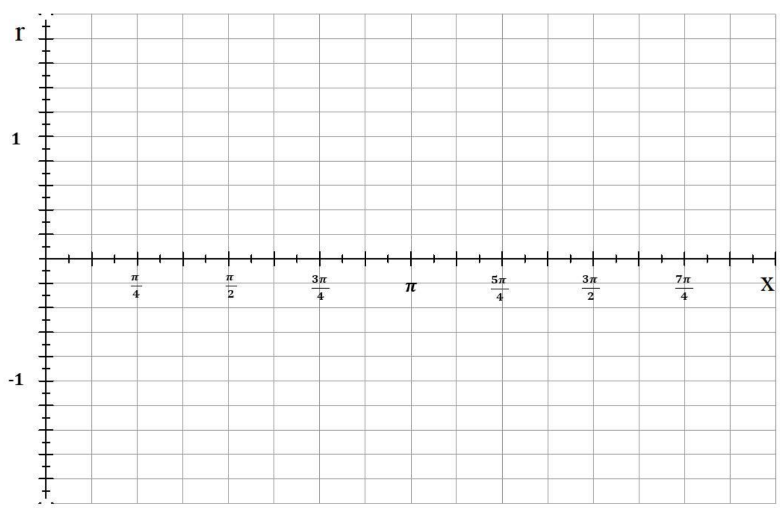
Problem 1. Consider the function $y = f(x) = \sin(x)\sin(x)$ in the interval $0 \leq x \leq 2\pi$. Let this function be Y_1 in your graphing calculator. Set your viewing window so that you have $0 \leq X \leq 2\pi$ XScl $= \pi/4$ and $-2 \leq Y \leq 2$ YScl $= 1$. (Make sure your calculator is in radian mode.)

Part (a). Sketch the graph of f on your calculator. On which subintervals is the output of f increasing?

Part (b). At which input values does the function f have a local minimum output?

Part (c). Keeping L_1 the same as in Problem 2 above, turn off Y_1 and have the calculator sketch the graphs of the five average rate of change functions determined by L_1.

Part (d). Using the average rate of change graphs as your guide, carefully sketch what you think should be the graph of the derivative function $r = f'(x)$ on the grid provided.



Part (e). Does your derivative graph correctly predict the subintervals where the output of f is increasing or decreasing?

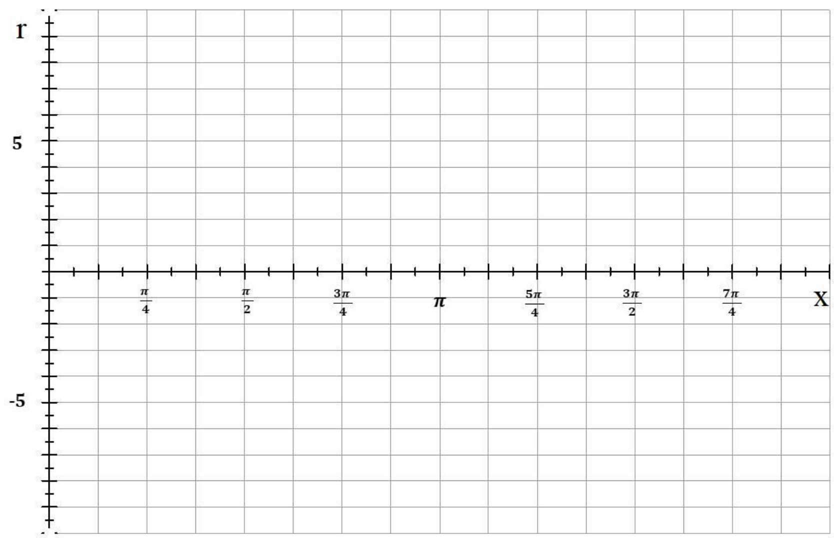
Problem 2. Consider the function $y = f(x) = \tan(x)$ in the interval $0 \leq x \leq 2\pi$. Let this function be Y_1 in your graphing calculator. Set your viewing window so that you have $0 \leq X \leq 2\pi$ XScl $= \pi/4$ and $-10 \leq Y \leq 10$ YScl $= 1$. (Put your calculator is in radian mode and DOT mode.)

Part (a). Sketch the graph of f on your calculator. On which subintervals is the output of f increasing?

Part (b). Does the function f have any local extrema?

Part (c). Keeping L_1 the same as in Problem 2 above, turn off Y_1 and have the calculator sketch the graphs of the five average rate of change functions determined by L_1.

Part (d). Using the average rate of change graphs as your guide, carefully sketch what you think should be the graph of the derivative function $r = f'(x)$ on the grid provided.



Part (e). Does your derivative graph correctly predict the subintervals where the output of f is increasing or decreasing?

Problem 3. In this investigation, we provided evidence that the derivative function for $f(x) = \ln(x)$ is the function $f'(x) = x^{-1}$ (when $x > 0$).

Part (a). At what input values will the instantaneous rate of change for f be exactly 2?

Part (b). Are there any input values where the derivative function has negative output?

Problem 4. Consider the function $y = f(x) = \ln(x)\sin(x)$ along with $r = g(x) = x^{-1}\cos(x)$.

Part (a). Consider the average rate of change function $r = \mathrm{ARC}_{0.0001}(x)$ for the function f. Explain why this function is a good approximation to the derivative function f'.

Part (b). Sketch the graph of the functions $\mathrm{ARC}_{0.0001}$ and g on the input interval $0 < x \leq 2\pi$. (Let $-2 \leq y \leq 2$). Based on these graphs, do you think that $f' = g$? Explain.

Problem 5. Consider the function $y = f(x) = \cos(x)\cos(x)$ along with $r = g(x) = \sin(x)\sin(x)$. Think about the average rate of change function $r = \mathrm{ARC}_{0.0001}(x)$ for the function f.

Sketch the graph of the functions $\mathrm{ARC}_{0.0001}$ and g on the input interval $0 < x \leq 2\pi$. (Let $-1 \leq y \leq 1$). Based on these graphs, do you think that $f' = g$? Explain.

Problem 6. Based on Problems 4 and 5, do you think that the following statement is true?

- The derivative function for the product of two functions is the product of the derivatives for these functions.

Explain your answer.

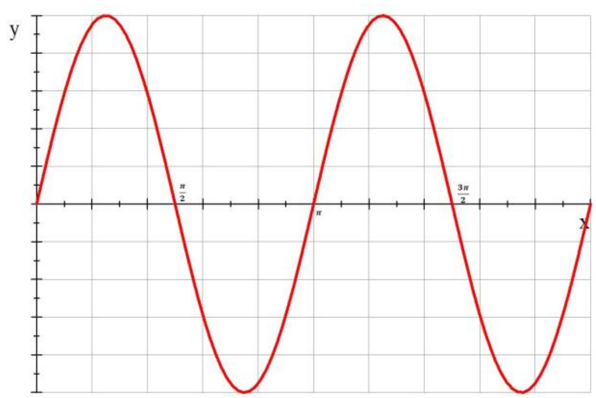
Answers to the Homework.

Problem 1.

Part (a). The graph of f is increasing on the input intervals $0 < x < \pi/2$ and $\pi < x < 3\pi/2$.

Part (b). The function f has local minimum output at the input values $x = 0$, $x = \pi$, and $x = 2\pi$.

Part (d).



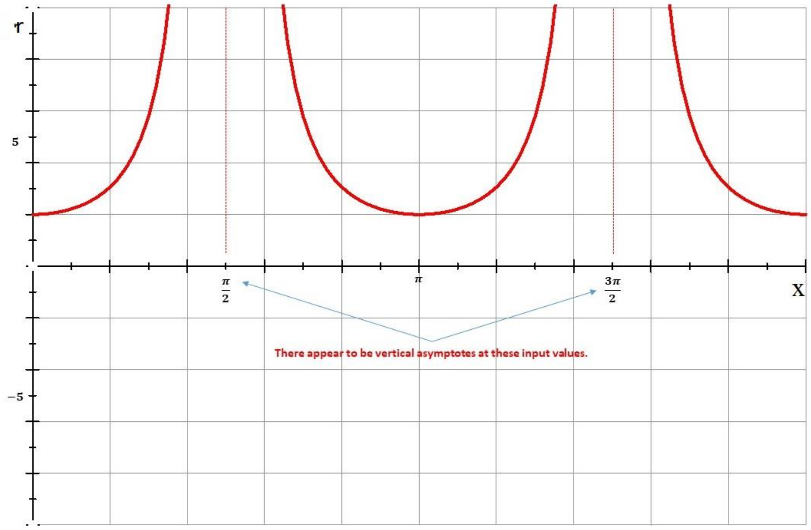
Part (e). The output of the derivative graph is positive on the input intervals where the graph of the function f is increasing, and the output of the derivative graph is negative on the input intervals where the graph of the function f is decreasing.

Problem 2.

Part (a). The tangent function is increasing on every input interval where it is defined.

Part (b). The tangent function has no local relative extrema.

Part (d).



Part (e). The output of the derivative function is always positive, and the graph of the tangent function is always increasing.

Problem 3.

Part (a).

$$f'(x) = 2 \quad \Rightarrow \quad \frac{1}{x} = 2 \quad \Rightarrow \quad x = \frac{1}{2}$$

Part (b). The derivative function will have negative output when $x < 0$; however, the derivative formula is only *valid* when $x > 0$ since the logarithm function is not defined when $x \leq 0$. Hence, the answer is "no." Note that this observation is consistent with the fact that the graph of the logarithm function is always increasing.

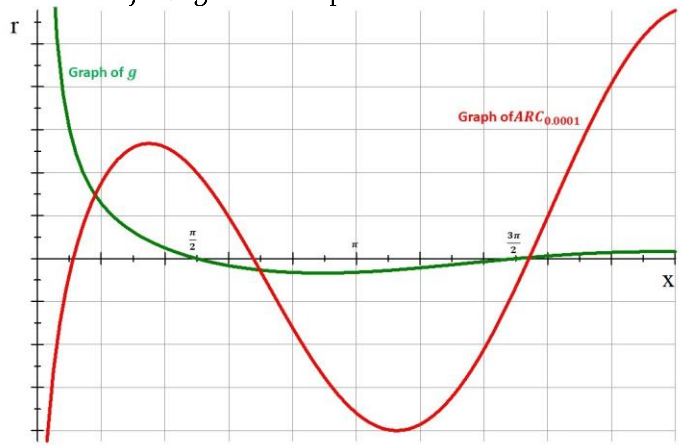
Problem 4.

Part (a). We know that the derivative function is defined by a limiting process. In particular, we know

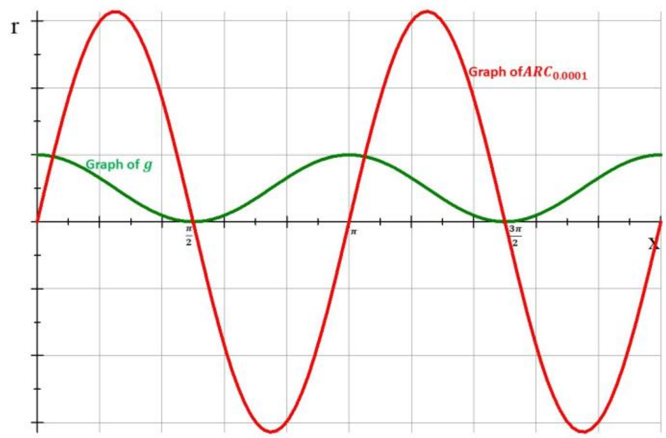
$$f'(x) = \lim_{h \to 0} \text{ARC}_h(x)$$

It therefore stands to reason that the functions ARC_h will serve as good approximations to f' when h is very close to 0.

Part (b). No, the graphs of these functions do not look very similar. Consequently, since we expect $\text{ARC}_{0.0001}$ to be a good approximation to f', the fact that the graph of $\text{ARC}_{0.0001}$ does not look like the graph of g is strong evidence that $f' \neq g$ on this input interval.



Problem 5. No, the graphs of these functions do not look very similar. Consequently, since we expect $\text{ARC}_{0.0001}$ to be a good approximation to f', the fact that the graph of $\text{ARC}_{0.0001}$ does not look like the graph of g is strong evidence that $f' \neq g$ on this input interval.



Problem 6. This statement appears to be *false* based on what we have observed in Problems 4 and 5.