In the previous investigation, we introduced the concept of the derivative function for a function $y = f(x)$ in an interval I. The derivative function is the end result of a limiting process in which we imagine average rate of change functions for f being constructed so that change in the output of f with respect to x is being measured over ever smaller subintervals of I. In symbols, the derivative function is defined by

$$\frac{df}{dx}(x) = \lim_{h \to 0} \mathrm{ARC}_h(x)$$

In the previous investigation, we found that visualizing this limiting process graphically can sometimes give us insight into what the limiting derivative function might be. In this investigation, we will explore how algebraic manipulation of the average rate of change functions can sometimes lead to a precise formula.

Problem 1. Consider the function $u = f(a) = a^2$ in the interval $-\infty < a < \infty$.

Part (a). Explain what is happening in each of the steps below.

$$\mathrm{ARC}_h(a) = \frac{f(a+h) - f(a)}{h} \qquad \text{Why?}$$

$$= \frac{(a+h)^2 - a^2}{h} \qquad \text{Why?}$$

$$= \frac{a^2 + 2ah + h^2 - a^2}{h} \qquad \text{Why?}$$

$$= \frac{2ah + h^2}{h} \qquad \text{Why?}$$

$$= \frac{h(2a+h)}{h} \qquad \text{Why?}$$

$$= 2a + h \quad (\text{As long as } h \neq 0) \quad \text{Why?}$$

Part (b). Explain why this algebraic manipulation allows us to conclude that $\frac{df}{da}(a) = 2a$.

Problem 2. Consider the function $y = f(x) = \sqrt{2}$ in the interval $-\infty < x < \infty$. This function produces the same output value for every input value and is therefore called a *constant output* function.

Part (a). Use algebraic manipulation to show that the average rate of change functions for the function f are given by the formula $r = \mathrm{ARC}_h(x) = 0$ (as long as $h \neq 0$).

Part (b). What is the formula for the derivative function $r = f'(x)$? Explain your answer.

Problem 3. Consider the function $p = f(t) = t^{-1}$ in the interval $(-\infty, 0) \cup (0, \infty)$. (This function is not defined when $t = 0$.)

Part (a). Complete the following derivation by filling in the missing algebra steps.

$$r = \mathrm{ARC}_h(t) = \frac{f(t+h) - f(t)}{h}$$

$$= \frac{(t+h)^{-1} - t^{-1}}{h}$$

[Fill this in]

$$= \frac{1}{h} \cdot \frac{t - (t+h)}{t(t+h)}$$

[Fill this in]

$$= -\frac{1}{(t+h)t} \qquad \text{(As long as } h \neq 0\text{)}$$

Part (b). What is the formula for the derivative function $r = \frac{df}{dt}(t)$? Explain your answer.

Problem 4. Consider the function $y = f(x) = x^3$.

Part (a). Use algebra to show that the formula for the average of change function can be rewritten as

$$r = \mathrm{ARC}_h(x) = 3x^2 + 3xh + h^2 \qquad \text{(As long as } h \neq 0\text{)}$$

Part (b). What is the formula for the derivative function $r = f'(x)$? Explain your answer.

Problem 6. Consider the function $y = f(m) = \sqrt{m}$ in the interval $0 \le x < \infty$.

Part (a). Simplify the expression $\left(\sqrt{m+h} - \sqrt{m}\right) \cdot \left(\sqrt{m+h} + \sqrt{m}\right)$ as much as possible.

Part (b). Evaluate the limiting process $\lim_{h \to 0} \dfrac{1}{\sqrt{m+h} + \sqrt{m}}$. Explain your strategy.

Part (c). Explain what is going on in each step of the following derivation.

$$r = \mathrm{ARC}_h(m) = \frac{\sqrt{m+h} - \sqrt{m}}{h}$$

$$= \left(\frac{\sqrt{m+h} - \sqrt{m}}{h}\right) \cdot \left(\frac{\sqrt{m+h} + \sqrt{m}}{\sqrt{m+h} + \sqrt{m}}\right)$$ Why is this expression equal to the previous one?

$$= \frac{\left(\sqrt{m+h} - \sqrt{m}\right) \cdot \left(\sqrt{m+h} + \sqrt{m}\right)}{h \cdot \left(\sqrt{m+h} + \sqrt{m}\right)}$$ Why is this expression equal to the previous one?

$$= \frac{h}{h \cdot \left(\sqrt{m+h} + \sqrt{m}\right)}$$ Why is this expression equal to the previous one?

$$= \frac{1}{\left(\sqrt{m+h} + \sqrt{m}\right)}$$ (As long as $h \neq 0$) Why is this expression equal to the previous one?

Part (d). What is the formula for the derivative function $r = \frac{df}{dm}(m)$? Explain your answer.

Problem 7. Use algebraic manipulation on the average rate of change functions to determine the formula for $r = f'(p)$ if the function f is defined by $s = f(p) = p^2 - 2p$ in the interval $-\infty < p < \infty$.

Homework.

Problem 1. Consider the function $y = f(x) = c$ in the interval $-\infty < x < \infty$, where c is a fixed real number. Use algebraic manipulation on the average rate of change functions to show that $f'(x) = 0$.

Problem 2. Consider the function $y = f(t) = ct$ in the interval $-\infty < t < \infty$, where c is a fixed real number. Use algebraic manipulation on the average rate of change functions to show that $f'(t) = c$.

Problem 3. Consider the function $z = f(q) = 4q^2 - 3q + 5$ in the interval $-\infty < q < \infty$.

Part (a). Use algebraic manipulation to show that the average rate of change functions for f are given by the formula

$$\mathrm{ARC}_h(q) = 8q - 3 + 4h \qquad \text{(As long as } h \neq 0\text{)}$$

Part (b). What does this tell you about the derivative function $r = f'(q)$?

Problem 4. Consider the function $y = f(x) = (5x - 1)^{-1}$ in the interval $\left(-\infty, {}^{1}/_{5}\right) \cup \left({}^{1}/_{5}, \infty\right)$.

Part (a). Use algebraic manipulation to show that the average rate of change functions for f are given by the formula

$$\text{ARC}_h(x) = -\frac{5}{(5x+5h-1)(5x-1)} \qquad \text{(As long as } h \neq 0\text{)}$$

Part (b). What does this tell you about the derivative function $r = f'(x)$?

Problem 5. Consider the function $t = f(w) = w^{-2}$ in the interval $(-\infty, 0) \cup (0, \infty)$. Use algebraic manipulation on the average rate of change functions to show that

$$\frac{df}{dw}(w) = -\frac{2}{w^3}$$

Problem 6. Consider the function $v = g(z) = 3z^2 - 2z + 4$ in the interval $-\infty < z < \infty$. Use algebraic manipulation on the average rate of change functions to determine the formula for the derivative function $r = g'(z)$.

Problem 7. Consider the function $m = g(n) = (2n-5)^2$ in the interval $-\infty < n < \infty$. Use algebraic manipulation on the average rate of change functions to determine the formula for the derivative function $r = \frac{dg}{dn}(n)$.

Answers to the Homework.

Problem 1. We begin by constructing and simplifying the average rate of change functions for f. Observe

$$\text{ARC}_h(x) = \frac{f(x+h) - f(x)}{h} = \frac{c - c}{h} = 0 \qquad \text{(As long as } h \neq 0\text{)}$$

Now, based on this information, we may conclude

$$f'(x) = \lim_{h \to 0} \text{ARC}_h(x) = \lim_{h \to 0} 0 = 0$$

Problem 2. We begin by constructing and simplifying the average rate of change functions for f. Observe

$$\text{ARC}_h(t) = \frac{f(t+h) - f(t)}{h} = \frac{c(t+h) - ct}{h} = \frac{ch}{h} = c \qquad \text{(As long as } h \neq 0\text{)}$$

Now, based on this information, we may conclude

$$f'(t) = \lim_{h \to 0} \text{ARC}_h(t) = \lim_{h \to 0} c = c$$

Problem 3.

Part (a). Observe

$$\begin{aligned}
\mathrm{ARC}_h(q) &= \frac{f(q+h)-f(q)}{h} \\
&= \frac{[4(q+h)^2-3(q+h)+5]-[4q^2-3q+5]}{h} \\
&= \frac{[4(q^2+2qh+h^2)-3(q+h)+5]-[4q^2-3q+5]}{h} \\
&= \frac{8qh+4h^2-3h}{h} \\
&= 8q+4h-3 \qquad \text{(As long as } h \neq 0)
\end{aligned}$$

Part (b). Observe

$$f'(q) = \lim_{h\to 0} \mathrm{ARC}_h\,(q) = \lim_{h\to 0}(8q+4h-3) = 8q-3$$

Problem 4. Consider the function $y = f(x) = (5x-1)^{-1}$ in the interval $\left(-\infty, {}^{1}/_{5}\right) \cup \left({}^{1}/_{5}, \infty\right)$.

Part (a). Observe

$$\begin{aligned}
\mathrm{ARC}_h(x) &= \frac{f(x+h)-f(x)}{h} \\
&= \frac{(5(x+h)-1)^{-1}-(5x-1)^{-1}}{h} \\
&= \frac{1}{h}\cdot\left[\frac{1}{5x+5h-1}-\frac{1}{5x-1}\right] \\
&= \frac{1}{h}\cdot\left[\frac{1}{5x+5h-1}\left(\frac{5x-1}{5x-1}\right)-\frac{1}{5x-1}\left(\frac{5x+5h-1}{5x+5h-1}\right)\right] \\
&= \frac{1}{h}\cdot\left[\frac{5x-1-(5x+5h-1)}{(5x+5h-1)(5x-1)}\right] \\
&= \frac{1}{h}\cdot\frac{-5h}{(5x+5h-1)(5x-1)} \\
&= -\frac{5}{(5x+5h-1)(5x-1)} \qquad \text{(As long as } h \neq 0)
\end{aligned}$$

Part (b). Based on the formula from Part (a), we know

$$f'(x) = \lim_{h\to 0} \mathrm{ARC}_h\,(x) = \lim_{h\to 0}\left(-\frac{5}{(5x+5h-1)(5x-1)}\right) = -\frac{5}{(5x-1)(5x-1)} = -\frac{5}{(5x-1)^2}$$

Problem 5. Observe that

$$\begin{aligned}
\mathrm{ARC}_h(w) &= \frac{f(w+h)-f(w)}{h} \\
&= \frac{(w+h)^{-2}-w^{-2}}{h} \\
&= \frac{1}{h}\cdot\left[\frac{1}{(w+h)^2}-\frac{1}{w^2}\right] \\
&= \frac{1}{h}\cdot\left[\frac{1}{(w+h)^2}\left(\frac{w^2}{w^2}\right)-\frac{1}{w^2}\left(\frac{(w+h)^2}{(w+h)^2}\right)\right] \\
&= \frac{1}{h}\cdot\left[\frac{w^2-(w+h)^2}{w^2(w+h)^2}\right] \\
&= \frac{1}{h}\cdot\left[\frac{w^2-(w^2-2wh+h^2)}{w^2(w+h)^2}\right] \\
&= \frac{1}{h}\cdot\left[\frac{2wh+h^2}{w^2(w+h)^2}\right] \\
&= -\frac{h+2w}{w^2(w+h)^2} \qquad \text{(As long as } h \neq 0\text{)}
\end{aligned}$$

Now, based on this derivation, we can conclude that

$$f'(w) = \lim_{h\to 0}\left[-\frac{h+2w}{w^2(w+h)^2}\right] = -\frac{2w}{w^2\cdot w^2} = -\frac{2}{w^3}$$

Problem 6. Observe that

$$\begin{aligned}
\mathrm{ARC}_h(z) &= \frac{g(z+h)-g(z)}{h} \\
&= \frac{[3(z+h)^2-2(z+h)+4]-[3z^2-2z+4]}{h} \\
&= \frac{[3z^2+6zh+3h^2-2z-2h+4]-[3z^2-2z+4]}{h} \\
&= \frac{6zh-2h+3h^2}{h} \\
&= 6z-2+3h \qquad \text{(As long as } h \neq 0\text{)}
\end{aligned}$$

Based on this derivation, we may now conclude that

$$g'(z) = \lim_{h \to 0}(6z - 2 + 3h) = 6z - 2$$

Problem 7. Consider the function $m = g(n) = (2n - 5)^2$ in the interval $-\infty < n < \infty$. Use algebraic manipulation on the average rate of change functions to determine the formula for the derivative function $r = \frac{dg}{dn}(n)$.

$$\begin{aligned}
\text{ARC}_h(zn) &= \frac{g(n+h) - g(n)}{h} \\
&= \frac{(2(n+h) - 5)^2 - (2n-5)^2}{h} \\
&= \frac{(2n + 2h - 5)^2 - (2n-5)^2}{h} \\
&= \frac{[4n^2 + 8nh - 20n - 20h + 4h^2 + 25] - [4n^2 - 20n + 25]}{h} \\
&= \frac{8nh - 20h - 4h^2}{h} \\
&= 8n - 20 - 4h \qquad \text{(As long as } h \neq 0\text{)}
\end{aligned}$$

Based on this derivation, we may now conclude that

$$g'(n) = \lim_{h \to 0}(8n - 20 - 4h) = 8n - 20$$