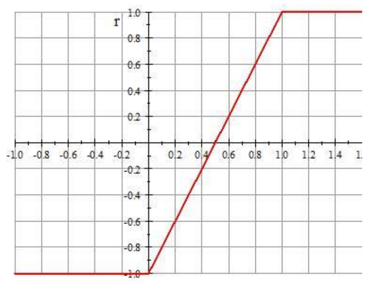
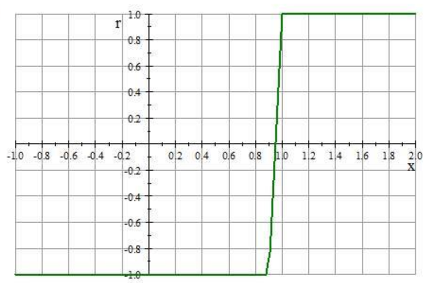
In this investigation, we will consider the possibility that the derivative function may not be defined for some input values of f.

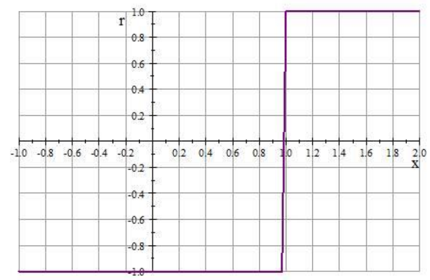
Problem 1. The diagram below shows the graphs of three average rate of change functions for the function $y = f(x) = 1 + |x - 1|$ in the interval $-1 \le x \le 2$.



Graph of ARC_1



Graph of $\text{ARC}_{0.1}$



Graph of $\text{ARC}_{0.01}$

Part (a). Based on these graphs, what do you think the value of $f'(1.6)$ should be? Explain your thinking.

Part (b). Based on these graphs, what do you think the value of $f'(0.8)$ should be? Explain your thinking.

Part (c). Based on these graphs, what do you think the value of $f'(1)$ should be? Explain your thinking.

Problem 2. Consider the function $y = f(x) = 1 + |x - 1|$ from Problem 1.

Part (a). Use algebra to show that

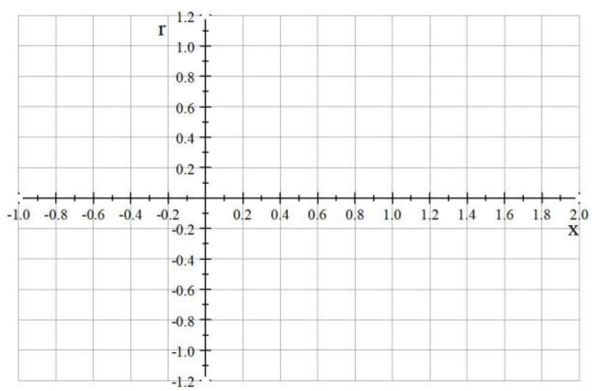
$$\text{ARC}_h(1) = \frac{|h|}{h} = \begin{cases} 1 & \text{if} \quad h > 0 \\ -1 & \text{if} \quad h < 0 \end{cases}$$

Part (b). Think about the formula in Part (a). If we consider only *positive* values of h, what can we say about the value of $\text{ARC}_h(1)$ as the values of h get closer and closer to 0? What would you predict $f'(1)$ should be?

Part (c). Think about the formula in Part (a). If we consider only *negative* values of h, what can we say about the value of $\text{ARC}_h(1)$ as the values of h get closer and closer to 0? What would you predict $f'(1)$ should be?

Part (d). Explain why we must conclude that $f'(1)$ cannot be defined.

Part (e). Sketch a trace of the graph of the derivative function $r = f'(x)$ for the function f. What strategy did you use?



Differentiable Function

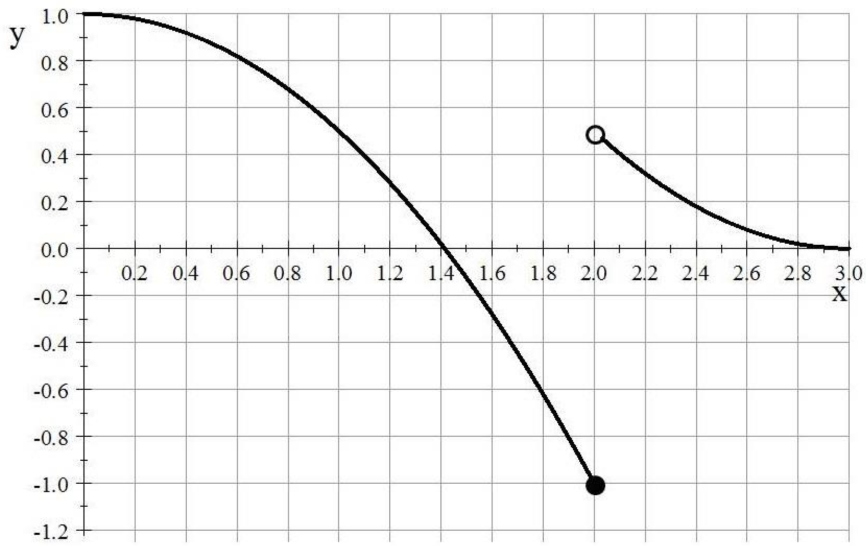
- Let $y = f(x)$ be a function. We say that f is *differentiable* at an input value $x = a$ provided $f'(a)$ exists.
- We say that f is *differentiable* on a set of input values provided f is differentiable at every input value in the set.

Problem 4. Let $y = f(x)$ be a function in an interval I.

Part (a). If f is undefined at an input value $x = a$, is it possible for $r = ARC_h(a)$ to be defined at $x = a$? Explain.

Part (b). If f is undefined at an input value $x = a$, is it possible for f to be differentiable at $x = a$? Explain.

Problem 5. The diagram below shows the graph of a function $y = f(x)$ that has a jump in its output at the input value $x = 2$.



Part (a). Use this trace to estimate the average rate of change in the output of f with respect to x over the intervals [2,3], [2,2.2], [2,2.1], [2,2.05], and [2,2.01].

$\text{ARC}_1(2) \approx$ $\qquad$ $\text{ARC}_{0.2}(2) \approx$ $\qquad$ $\text{ARC}_{0.1}(2) \approx$ $\qquad$ $\text{ARC}_{0.05}(2) \approx$ $\qquad$ $\text{ARC}_{0.01}(2) \approx$

Part (b). What pattern do you see? What does this pattern tell you about whether f is differentiable at $x = 2$?

The reason the function f in Problem 5 fails to be differentiable at the input value $x = 2$ is because

$$\lim_{\substack{h \to 0 \\ (\text{Keeping } h>0)}} f(2+h) = 5$$

This causes a problem because, as a result, $\Delta_y\big(f(2), f(2+h)\big)$ gets closer and closer to 6 as positive values of h get closer and closer to 0. Consequently, the values of the fraction

$$\text{ARC}_h(2) = \frac{\Delta_y\big(f(2), f(2+h)\big)}{h}$$

grow larger and larger without bound; and the limiting process produces no value. This leads us to the following important conclusion, one we will use in later investigations.

Differentiability Implies Continuity (See Investigation 3)

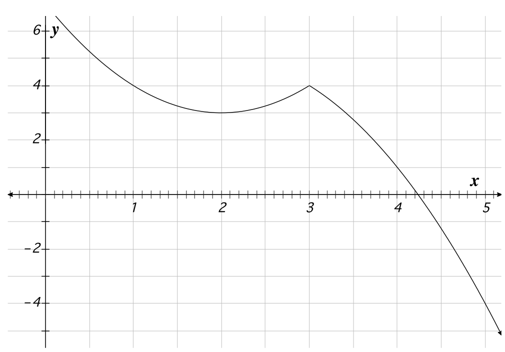
If $y = f(x)$ is differentiable at an input value $x = a$, then f is *continuous* at $x = a$; that is, the output value $f(a)$ exists, and

$$\lim_{h \to 0}\big(f(a+h)\big) = f(a)$$

Homework.

Problem 1. Consider the function $y = f(x)$ defined by the formula

$$f(x) = \begin{cases} 3 + (x-2)^2 & \text{if} \quad x < 3 \\ 5 - (x-2)^2 & \text{if} \quad 3 \le x \end{cases}$$



Part (a). Use algebra to show that, if $h > 0$, then the average rate of change for f on the input interval from $x = 3$ to $x = 3 + h$ is given by $\text{ARC}_h(3) = -2 - h$.

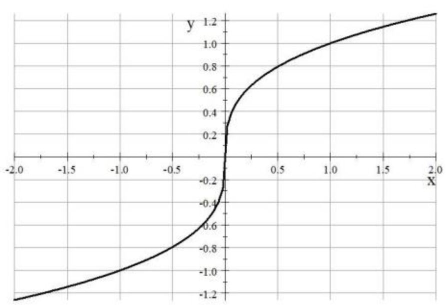
Part (b). If $h < 0$, what is the formula for the average rate of change for f on the input interval from $x = 3$ to $x = 3 + h$?

Part (c). Consider the limiting process $\lim_{h \to 0} \text{ARC}_h(3)$.

- If we restrict this process to considering only $h > 0$, what does the process yield?
- If we restrict this process to considering only $h < 0$, what does the process yield?

Part (d). Is the function f differentiable at $x = 3$? Justify your answer.

Problem 2. Consider the function $y = f(x) = \sqrt[3]{x}$. Here is a graph of this function.



Part (a). Use algebra to show that

$$\text{ARC}_h(0) = \frac{1}{\sqrt[3]{h^2}}$$

Part (b). Consider the limiting process $\lim_{h \to 0} \text{ARC}_h(0)$. Does this process produce a value?

Part (c). Is the function f differentiable at $x = 0$? Justify your answer.

Problem 3. Consider once again the function $y = f(x)$ defined by the formula

$$f(x) = \begin{cases} 3 + (x-2)^2 & \text{if} \quad x < 3 \\ 5 - (x-2)^2 & \text{if} \quad 3 \le x \end{cases}$$

Part (a). Suppose that $a < 3$, and let $h < \Delta_x(a, 3)$. (In other words, the value of h is less than the distance from a to 3.) Explain why we know $a + h < 3$.

Part (b). Show that, under the assumption that $h < \Delta_x(a, 3)$, we have

$$\text{ARC}_h(a) = 2a - 4 + h \quad (h \neq 0)$$

Part (c). Is the function f differentiable at $x = a$?

Answers to the Homework.

Problem 1.

Part (a). If $h > 0$, then $3 + h > 3$; and we know

$$\begin{aligned}
\text{ARC}_h(3) &= \frac{f(3+h) - f(3)}{h} \\
&= \frac{[5 - (1+h)^2] - 4}{h} \\
&= \frac{1 - (1 + 2h + h^2)}{h} \\
&= \frac{-2h - h^2}{h} \\
&= -2 - h \qquad (h \neq 0)
\end{aligned}$$

Part (b). If $h < 0$, then $3 + h < 3$; and we know

$$\begin{aligned}
\text{ARC}_h(3) &= \frac{f(3+h) - f(3)}{h} \\
&= \frac{[3 + (1+h)^2] - 4}{h} \\
&= \frac{1 + (1 + 2h + h^2)}{h} \\
&= \frac{2h + h^2}{h} \\
&= 2 + h \qquad (h \neq 0)
\end{aligned}$$

Part (c). Based on Parts (a) and (b), we know

$$\lim_{\substack{h \to 0 \\ (\text{Keeping } h>0)}} \text{ARC}_h(3) = -2 \qquad\qquad \lim_{\substack{h \to 0 \\ (\text{Keeping } h<0)}} ARC_h(3) = 2$$

Part (d). Since the limiting process produces different results depending on how we let the values of h approach 0, the function f is not differentiable at the input value $x = 3$.

Problem 2.

Part (a). Observe

$$\text{ARC}_h(0) = \frac{f(0+h) - f(0)}{h} = \frac{\sqrt[3]{h}}{h} = h^{1/3-1} = h^{-2/3} = \frac{1}{\sqrt[3]{h^2}}$$

Part (c). If we apply the limiting process to this expression, we find that it produces no real number value. In particular,

$$\lim_{\substack{h\to 0 \\ (\text{Keeping } h>0)}} \mathrm{ARC}_h\,(0) = +\infty$$

Part (d). The function f is not differentiable at the input value $x = 0$.

Problem 3.

Part (a). Since $h < 3 - a$, we know that $a + h < a + (3 - a) = 3$. Consequently, when $h < 3 - a$, we know that $f(a + h) = 3 + (x - 2 + h)^2$.

Part (b). We know that

$$\begin{aligned}
\mathrm{ARC}_h(a) &= \frac{f(a+h) - f(a)}{h} \\
&= \frac{\left[3 + (a - 2 + h)^2 - [3 + (a-2)^2]\right]}{h} \\
&= \frac{[a^2 + 2ah - 4a - 4h + h^2 + 4] - [a^2 - 4a + 4]}{h} \\
&= \frac{2ah - 4h + h^2}{h} \\
&= 2a - 4 + h \qquad (h \neq 0)
\end{aligned}$$

Part (c). Yes, the function is differentiable at this input value. We know this because the limiting process involves allowing the values of h to get closer and closer to 0. Therefore, in the limiting process, we can assume $h < \Delta_x(a, 3)$. Indeed, we can assume the values of h are smaller than *any* positive number we choose. Thus, we know

$$\lim_{h\to 0} \mathrm{ARC}_h\,(a) = \lim_{\substack{h\to 0 \\ (\text{Keeping } h<\Delta_x(a,3))}} \mathrm{ARC}_h(a) = 2a - 4$$