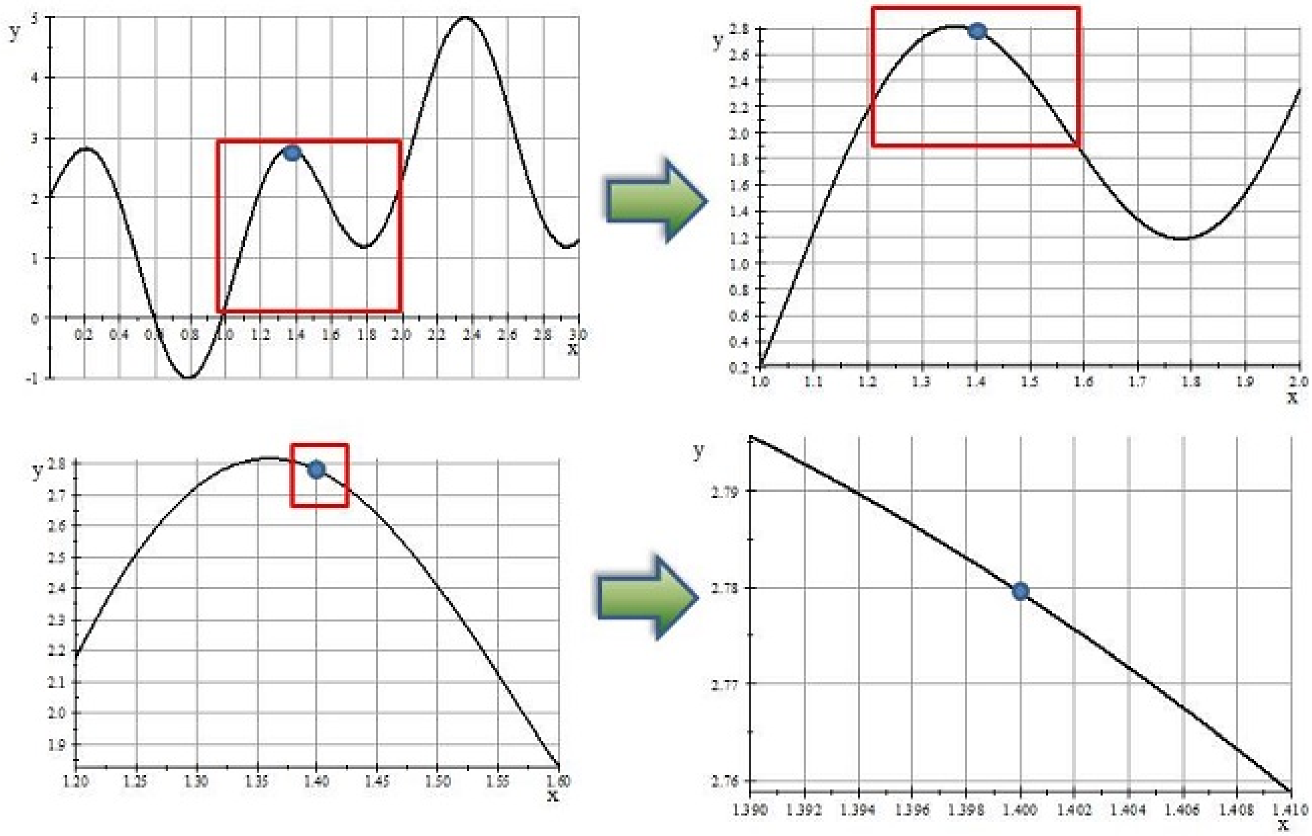
INTRODUCING THE DERIVATIVE

We say that a function f is *differentiable* (or *locally linear*) at a fixed input value $x = a$ in the domain of f provided, when we zoom in on the graph of f near the point $(a, f(a))$, the graph starts to look more and more like a straight line passing through the point $(a.f(a))$. For example, consider the graph of a function f shown below, along with the fixed point $(1.4, 2.77)$ on the graph.



As we zoom in repeatedly on the portion of the graph near the point $(1.4, 2.77)$, we see that the graph starts to look like a straight line.

- If a function f has a discontinuity at an input value $x = a$, then f cannot be locally linear at $x = a$.
- If a function f is continuous at $x = a$, but the graph has a kink or corner at $(a, f(a))$, then f cannot be locally linear at $x = a$.

If a function f is locally linear at an input value $x = a$, then the graph of f looks more and more like a straight line passing through $(a, f(a))$ as we zoom in on the graph near that point. The slope of this straight line is called the *derivative* of the function f at the input value $x = a$. The straight line is called the *tangent line* to the graph of f at the point $(a, f(a))$.

- It is customary to use the symbol $f'(a)$ to represent the derivative of the function f at the input value $x = a$.

- Using the point-slope formula, the tangent line to the graph of f at the point $(a, f(a))$ is always given by the formula $y - f(a) = f'(a)[x - a]$.

Problem 1. Using the graph in the last zoom box for the function f above, estimate the value of $f'(1.4)$. Use your estimate to construct the approximate formula for the tangent line to the graph of f at the point $(1.4, 2.77)$.

Problem 2. Suppose that f is a differentiable function at the input value $x = 1$. If $f(1) = -3$ and $f'(1) = 2.5$, what is the slope-intercept formula for the tangent line to the graph of f at the point $(1, f(1))$?

Problem 3. Suppose that f is a differentiable function at the input value $x = 4$ and suppose the slope-intercept formula for the tangent line to the graph of f at the point $(4, f(4))$ is given by

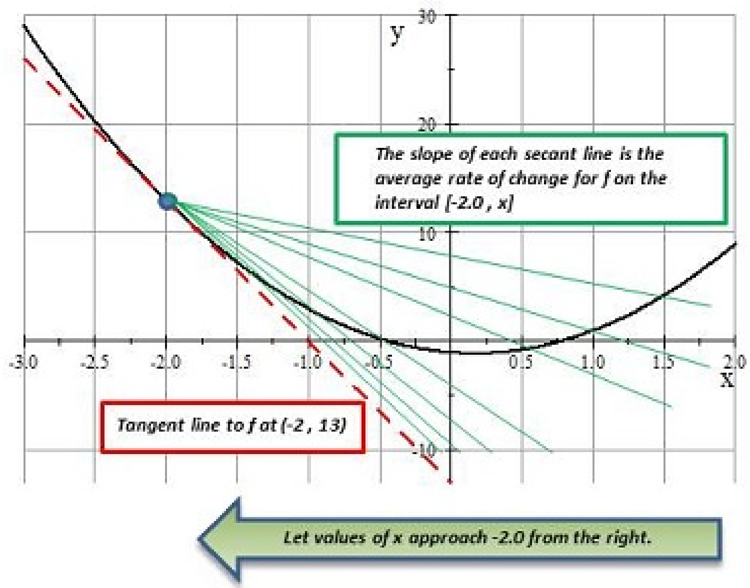
$$y = -3.25x + 9$$

Part (a): What is the value of $f'(4)$?

Part (b): What is the value of $f(4)$?

HOMEWORK: Section 2.7 Page 149 Problems 20 and 21.

If f is a differentiable function at the input value $x = a$, then $f'(a)$ is the limit of the average rates of change for f on the intervals $[x, a]$ (for $x < a$) and $[a, x]$ (for $x > a$).



This tells us that we can compute the value of $f'(a)$ using a limiting process. In particular, we know

$$f'(a) = \lim_{x \longrightarrow a} \frac{f(x) - f(a)}{x - a}$$

Therefore, as long as the average rate of change function

$$A_f(x) = \frac{f(x) - f(a)}{x - a}$$

has a removable discontinuity at the input value $x = a$, the function f will be differentiable at $x = a$.

Example 1 *Tom tosses a rock vertically upward and then steps out of the way until the rock hits the ground. If the vertical height in feet of the rock above the ground since Tom tossed it is given by the function* $s(t) = -32t^2 + 100t + 4$, *(where t represents the number of seconds since Tom tossed the rock) what is the instantaneous velocity of the rock two seconds after Tom tossed it?*

Solution. The instantaneous velocity of the rock at the input value $t = 2$ seconds is the limit of the average velocities of the rock as the values of t get closer and closer to 3. In other words, the velocity will be $s'(2)$. Now, the average velocity for the rock on any time interval $[t, 2]$ or $[2, t]$ is given by the formula

$$A_s(t) = \frac{s(t) - s(2)}{t - 2} = \frac{\left[-32t^2 + 100t + 4\right] - 76}{t - 2} = \frac{-32t^2 + 100t - 72}{t - 2}$$

Notice that the average velocity function has a discontinuity at $t = 2$ seconds, because inputting this value of t results in division by 0. Now, $t = 2$ is also a root of the numerator; this means that $t - 2$ must be a factor of the numerator. In fact, with a little effort, we see that

$$A_s(t) = \frac{(t - 2)(-32t + 36)}{t - 2} = 36 - 32t \quad \text{(as long as } t \neq 2\text{)}$$

Therefore, the average velocity function has a removable discontinuity at $t = 2$, and we know

$$s'(2) = \lim_{t \longrightarrow 2} \frac{-32t^2 + 100t - 72}{t - 2} = \lim_{t \longrightarrow 2} (36 - 32t) = -28 \text{ feet per second}$$

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Problem 4. A particle is moving back and forth along a straight line, and its distance in feet from the starting point is given by $p(t) = 3t^2 - 8t + 2$, where t represents the number of minutes since the particle started moving. What is the instantaneous velocity of the particle when $t = 1$ minute?

Problem 5. Find the formula for the tangent line to the graph of the function $f(x) = x^2 - 3x + 1$ at the point $(2, f(2))$.

HOMEWORK: Section 2.6 Pages 148 - 149 Problems 13, 16, 18, and 19

Many people think it is more difficult to factor a polynomial than it is to expand a binomial raised to a power. If you are one of those people, there is a way to avoid obscure factoring when trying to compute the derivative for a function at a specified input value. The trick is to make a *variable substitution.* In particular, if we let $h = x - a$ in the limit formula for computing the derivative, we have

$$f'(a) \quad = \quad \underbrace{\lim_{x \longrightarrow a} \frac{f(x) - f(a)}{x - a}}_{\text{original definiiton}} \quad = \quad \underbrace{\lim_{h \longrightarrow 0} \frac{f(a+h) - f(a)}{h}}_{\text{Making the substituion } h = x - a}$$

Let's see how this change of variable affects the way we compute the instantaneous velocity from Example 1. Observe

$$\begin{aligned}
s'(2) &= \lim_{h \longrightarrow 0} \frac{s(2+h) - s(2)}{h} \\
&= \lim_{h \longrightarrow 0} \frac{\left[-32(2+h)^2 + 100(2+h) + 4\right] - 76}{h} \\
&= \lim_{h \longrightarrow 0} \frac{\left[-32(4 + 4h + h^2) + 100(2+h) + 4\right] - 76}{h} \\
&= \lim_{h \longrightarrow 0} \frac{\left[-128 - 128h - 32h^2 + 200 + 100h + 4\right] - 76}{h} \\
&= \lim_{h \longrightarrow 0} \frac{-32h^2 - 28h}{h} \\
&= \lim_{h \longrightarrow 0} \frac{h\,(-32h - 28)}{h} \\
&= \lim_{h \longrightarrow 0} (-32h - 28) \\
&= -28 \text{ feet per second}
\end{aligned}$$

Notice that we did not have to factor a polynomial in the process of computing the value of this limit. We did, however, have to go through a fair amount of algebra as we expanded powers and distributed. Some people prefer this over factoring.

There are occasions when the alternative formulation of the limit definition is the only practical way to compute the derivative, because factoring is difficult or even impossible. For the reason, the alternative formulation has become the standard.

Example 2 *Construct a function that gives the slope of the tangent line to the graph of $f(x) = \frac{1}{x}$ at any input value $x = a$ other than $x = 0$.*

Solution. We want a formula for the derivative $f'(a)$. This is an occasion where the alternative formulation comes in handy. Observe

$$\begin{aligned}
f'(a) &= \lim_{h \longrightarrow 0} \frac{f(a+h) - f(a)}{h} \\
&= \lim_{h \longrightarrow 0} \frac{1/(a+h) - 1/a}{h} \\
&= \lim_{h \longrightarrow 0} \frac{1}{h}\left(\frac{1}{a+h} - \frac{1}{a}\right) \\
&= \lim_{h \longrightarrow 0} \frac{1}{h}\left(\frac{a-(a+h)}{a(a+h)}\right) \\
&= \lim_{h \longrightarrow 0} \frac{1}{h}\left(\frac{-h}{a(a+h)}\right) \\
&= \lim_{h \longrightarrow 0} \frac{-1}{a(a+h)} \\
&= \frac{-1}{a^2}
\end{aligned}$$

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Problem 6. What is the slope-intercept formula for the tangent line to the graph of $f(x) = \frac{1}{x}$ at the point $(2, f(2))$?

Problem 7. What is the slope-intercept formula for the tangent line to the graph of $f(x) = \frac{1}{x}$ at the point $(-3, f(-3))$?

Problem 8. Construct a function that gives the slope of the tangent line to the graph of $f(x) = x^2$ at any input value $x = a$.

HOMEWORK: Section 2.7 Pages 148-150 Problems 9, 14, 15, 31, 34, 46, 47, 49