WHAT THE DERIVATIVE CAN TELL US

Let f be a function of the input variable x. We can define a new function f' of the same input variable x which, for each input value x, gives us the slope of the tangent line to the graph of f at the point $(x, f(x))$. This new function is called the *derivative function* for f and is defined by the limiting process

$$f'(x) = \lim_{h \longrightarrow 0} \frac{f(x+h) - f(x)}{h}$$

Example 1 *What is the formula for the derivative function for $f(x) = x^3$?*

Solution. We know that

$$\begin{aligned} f'(x) &= \lim_{h \longrightarrow 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \longrightarrow 0} \frac{(x+h)^3 - x^3}{h} \\ &= \lim_{h \longrightarrow 0} \frac{\left[x^3 + 3x^2h + 3xh^2 + h^3\right] - x^3}{h} \\ &= \lim_{h \longrightarrow 0} \frac{3x^2h + 3xh^2 + h^3}{h} \\ &= \lim_{h \longrightarrow 0} \frac{h\left(3x^2 + 3xh + h^2\right)}{h} \\ &= \lim_{h \longrightarrow 0} (3x^2 + 3xh + h^2) \\ &= 3x^2 \end{aligned}$$

In the derivation above, we used the fact that $(x+h)^3 = x^3 + 3x^2h + 3xh^2 + h^3$. This formula comes from the fact that

$$(x+h)^3 = (x+h)(x+h)^2 = (x+h)(x^2 + 2xh + h^2)$$

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Problem 1. What is the slope of the tangent line to the graph of $f(x) = x^3$ at the point $(3, f(3))$?

Problem 2. What is the slope-intercept formula for the tangent line to the graph of $f(x) = x^3$ at the point $(-2, f(-2))$?

Problem 3. Are there any points on the graph of $f(x) = x^3$ where the tangent line to the graph of f is horizontal?

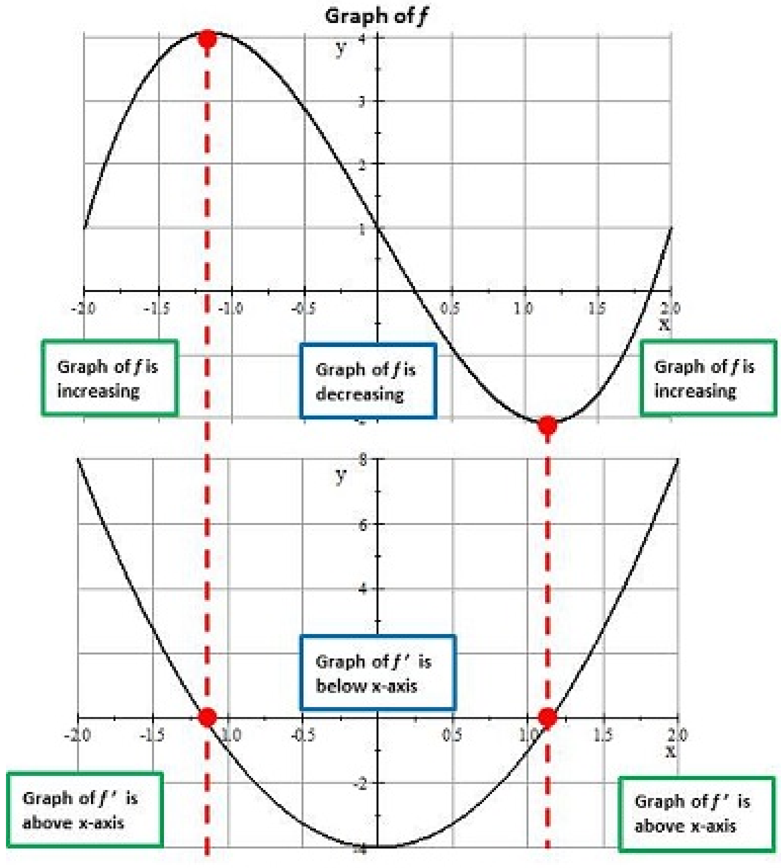
Problem 4. Using the limit definition for the derivative function, follow Example 1 above and determine the formula for f' when $f(x) = x^3 - 4x + 1$.

Problem 5. Use your derivative formula from Problem 4 to help construct the slope-intercept formula for the tangent line to the graph of $f(x) = x^3 - 4x + 1$ at the point $(2, f(2))$.

Problem 6. At what points on the graph of $f(x) = x^3 - 4x + 1$ will the tangent line be horizontal?

HOMEWORK: Section 2.8 Page 162 Problems 21, 23, 24, 25, 31

The diagram below shows the graph of the function $f(x) = x^3 - 4x + 1$ along with the graph of its derivative function. Take a look at how the behavior of one graph relates to the behavior of the other graph.



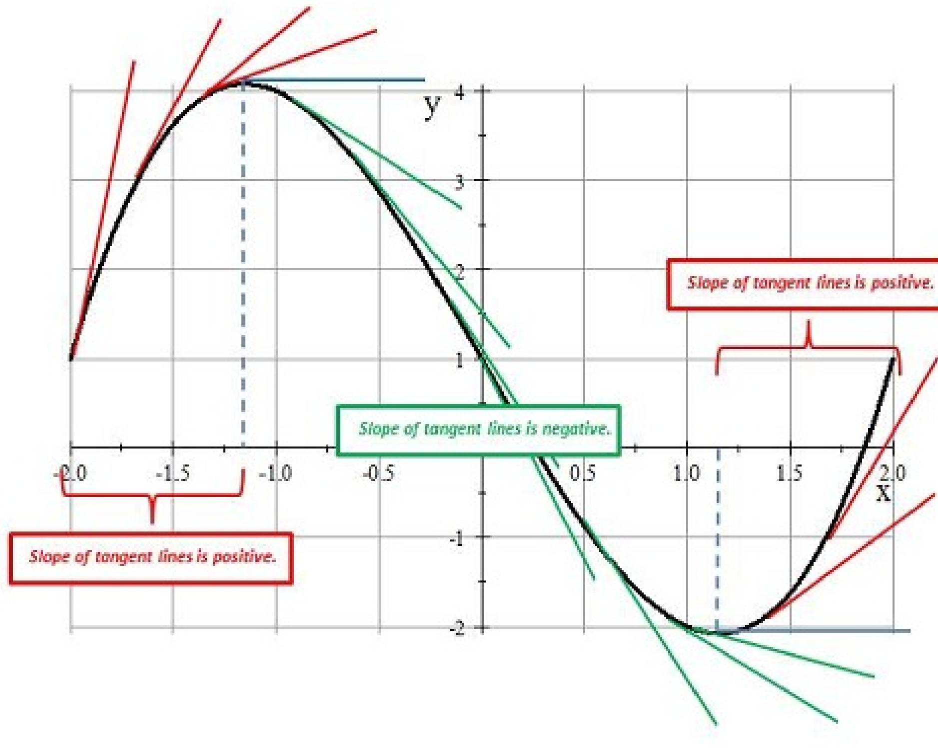
Graph of the Derivative Function for f

If we keep in mind that the derivative function for a function f gives the slope of tangent lines to the graph of f as its output, it is easy to see that relationships shown in the graphs above will always hold.

- If the graph of f has a turning point at $(a, f(a))$, then the tangent line to f at this point will be horizontal (have slope 0). Therefore, the graph of the derivative function f' must intersect the x-axis at $x = a$.

- If the graph of f is increasing (from left to right) on some interval $a < x < b$, then the tangent lines to the graph of f at any point $(x, f(x))$ will be increasing (and therefore have positive slope). Therefore, the the graph of the derivative function f' must be above the x-axis on this interval.

- If the graph of f is decreasing (from left to right) on some interval $a < x < b$, then the tangent lines to the graph of f at any point $(x, f(x))$ will be decreasing (and therefore have negative slope). Therefore, the the graph of the derivative function f' must be below the x-axis on this interval.

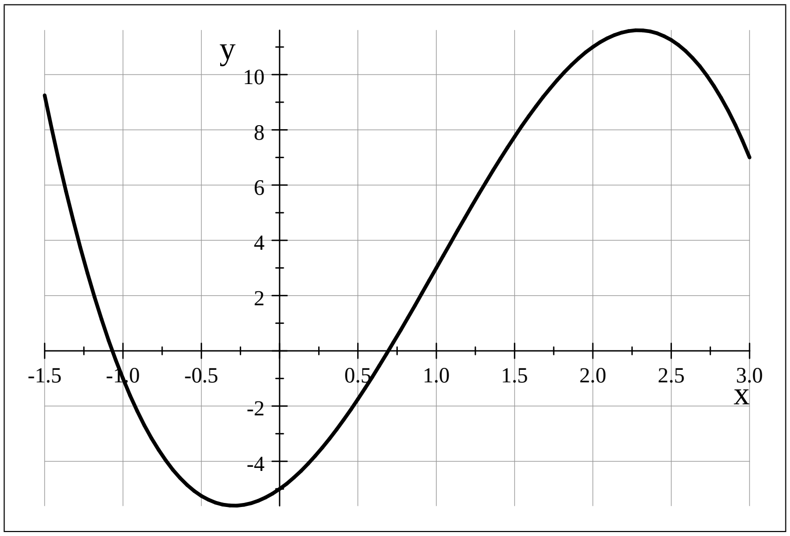


Problem 6. A point on the graph of a function f where the concavity changes is called an *inflection point* for the function f. Consider the graph of the function $f(x) = x^3 - 4x + 1$ shown in the figures above.

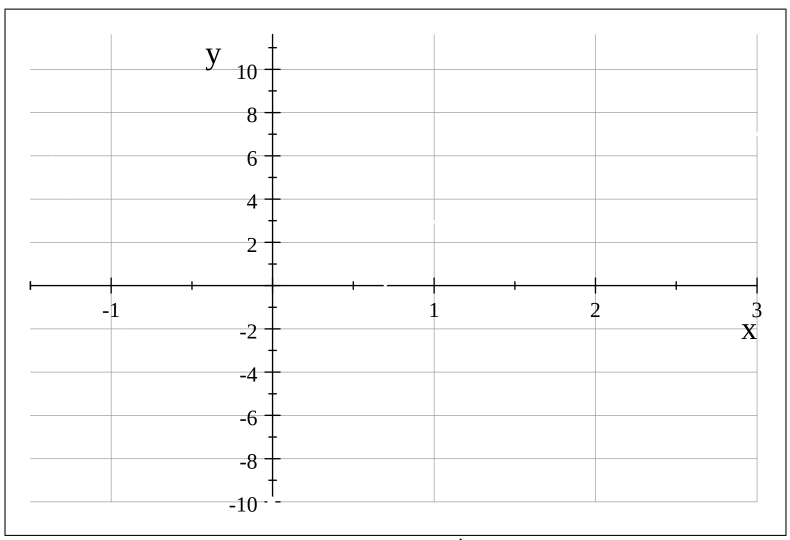
Part (a): Where does the function f have an inflection point?

Part (b): What can we say about the graph of the derivative function f' at this point?

Problem 7. The graph of a function f is shown below. On the grid provided, construct a rough sketch of the derivative function for f. You can sketch some tangent lines to the graph of f to help you. Be sure to indicate where the derivative function intersects the x-axis and where the derivative function will have a turning point.



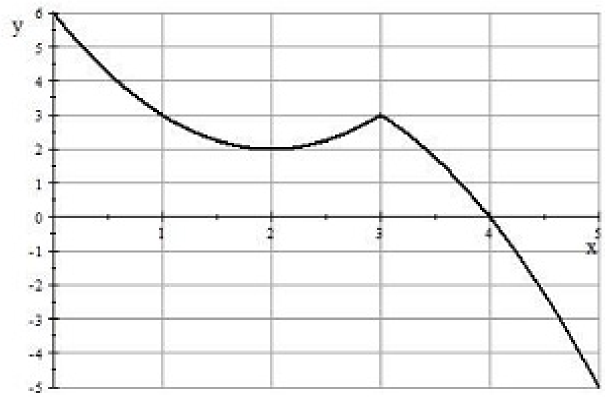
Graph of f



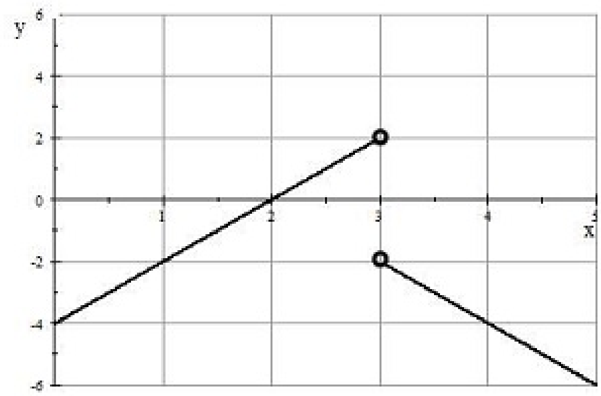
Graph of f'

HOMEWORK: Section 2.8 Pages 161 - 162 Problems 4, 5, 6, 10, 13, 14, 15

The diagram below shows the graph of a function f along with the graph of its derivative function.



Graph of f



Graph of f'

Problem 8. By sketching tangent lines to the graph of f, devise an explanation as to why the graph of f' looks the way it does.

HOMEWORK: Section 2.8 Page 161 Problems 3, 8, 9, 41, 42, 43, and 44

There is an alternative notation for the derivative which makes it easy to write down the specific derivative function for a given function f. This so-called *differential* notation, like the prime notation used above, date back to the eariliest days of calculus, long before the notion of limits had been developed:

The derivative function for a function f can be denoted by the symbol $\dfrac{df}{dx}$.

This notation is only a symbol, but we can often treat it like a fraction (especially in Calculus II). The "numerator" and "denominator" of this symbol are called *differentials.* In the early days of calculus, a "differential" was defined to be "a positive number so small that its square is equal to 0."

The reasoning behind this was simple: Suppose we want the slope of the tangent line to the function $f(x) = x^2$ at the point $(a, f(a))$. We cannot compute this directly from the definition of slope, since we have only one point on the line. We can approximate the slope very well by finding the slopes of secant lines between

$$(a, f(a)) \quad \text{and} \quad (a + da, f(a + da))$$

where da is an *extremely small* quantity. Note that the slope of the line between these two points is

$$\frac{\triangle f}{\triangle x} = \frac{f(a+da) - f(a)}{(a+da) - a} = \frac{(a+da)^2 - a^2}{da} = \frac{2a(da) + (da)^2}{da}$$

Today, what we would do next to compute the slope of the tangent line is simply factor the da term out of the numerator, cancel, and take the limit as da approaches 0:

$$f'(a) = \lim_{da \longrightarrow 0} \frac{2a(da) + (da)^2}{da} = \lim_{da \longrightarrow 0} (2a + da) = 2a$$

In the early days of calculus, however, there was no concept of the limit process. Early users of calculus did not consider finding the slope as a process, but rather as an algebra problem. They determined the slope by a sleight of hand — they *assumed* that the number da was *so small* that its square (or any other higher power) was equal to 0. Doing so, they found the slope of the tangent line

$$\frac{df}{dx} = \frac{2a(da) + (da)^2}{da} = \frac{2a(da) + 0}{da} = \frac{2a(da)}{da} = 2a$$

They used the special notation $\frac{df}{dx}$ to indicate that they had performed this trick. Of course, this trick does not make sense from a modern mathematical perspective, since there is no positive number whose square is 0. In the early days, calculus had many critics precisely because of this "shady" way of computing the slope of tangent lines. The most vexing part of the controversy, however, was that the end result of this "shady" computation was always correct! Whenever experimentation or observation could be used, they always verified the formulas that this method came up with.

It took well over a century for mathematicians to devise a way around the "differential" computation techniques. Our modern notion of the limit process is the end result . The formal definition of the limit and the computational rules that can be derived from it finally placed the calculus on a firm theoretical footing and paved the way for great advances in the 1800's.

Although the concept of the limit process has eliminated the need for differentials, they continue to hang on in our notation because they work so well *as notation.* For example, we have shown that the derivative function for

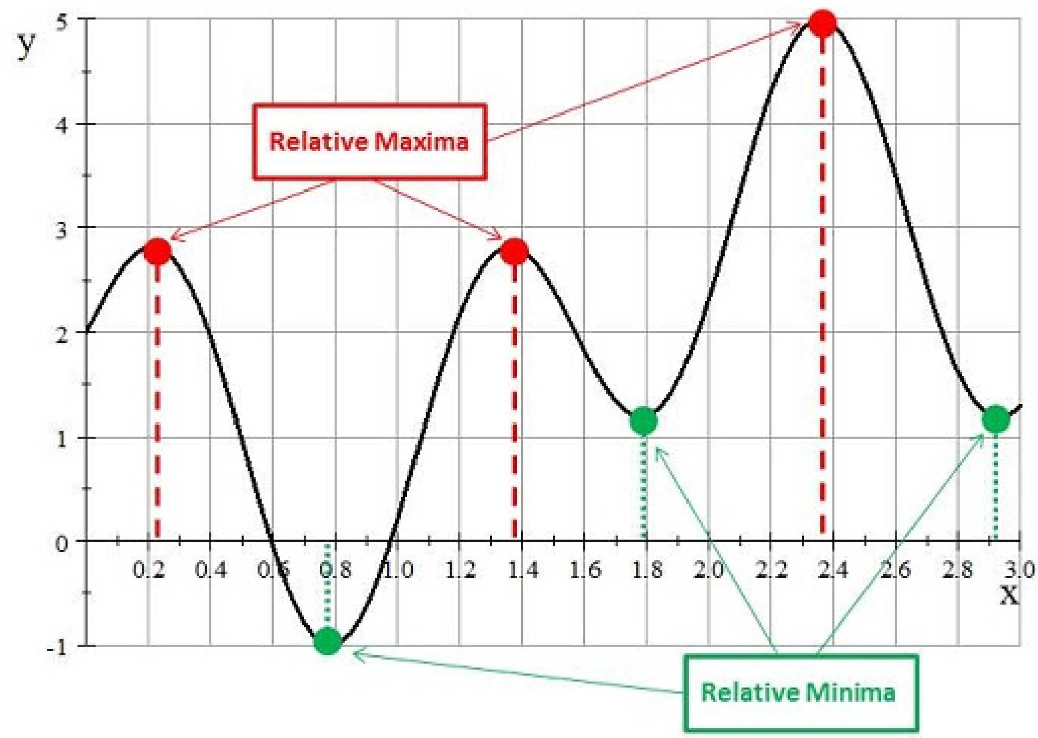
$$f(x) = x^3 \quad \text{is} \quad f'(x) = 3x^2$$

We can use differential notation to represent this fact in one compact equation:

$$\frac{d}{dx}\left[x^3\right] = 3x^2$$

We commonly write specific derivative formulas in this way.

A function $y = f(x)$ is said to have a *relative maximum output* at an input value $x = a$ provided $f(a)$ is the largest output for f on some small input interval containing $x = a$. We say that f has a *relative minimum output* at an input value $x = b$ provided $f(b)$ is the smallest output on some input interval containing $x = b$.

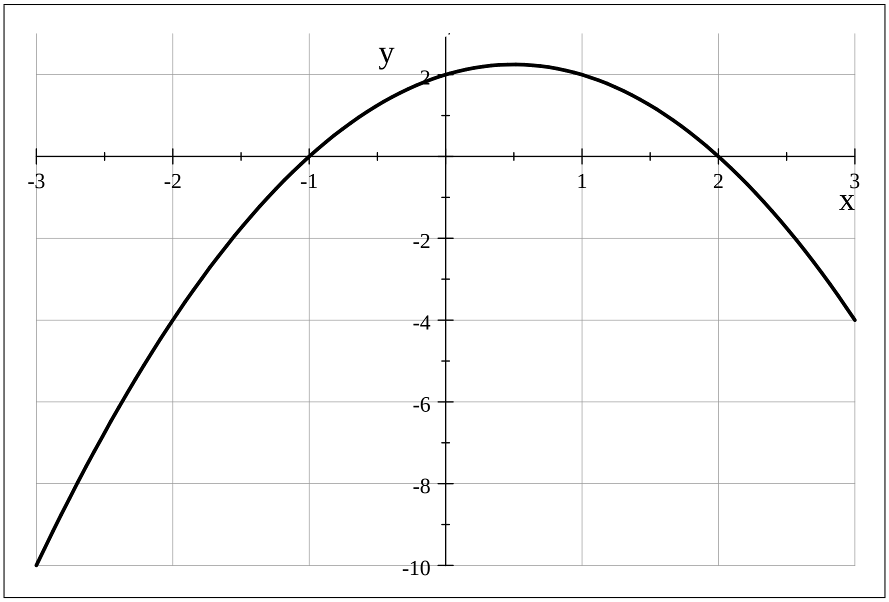


The function f whose graph is shown above has three relative maxima and three relative minima. The function has a relative maximum output of approximately 2.8 at the input value $x \approx 0.21$, another relative maximum output of approximately 2.8 at the input value $x \approx 1.39$, and a third relative maximum output of approximately 4.9 at the input value $x \approx 2.38$. This function has a relative minimum output of approximately -0.9 at the input value $x \approx 0.79$, another relative minimum output of approximately 1.3 at the input value $x \approx 1.8$, and a third relative minimum output of approximately 1.3 at the input value $x \approx 2.91$.

- If a function f has a relative maximum output at $x = a$, then the graph of f changes from increasing to decreasing at $x = a$. Therefore, the graph of f' changes from being above the x-axis to being below the x-axis at $x = a$.

- If a function f has a relative minimum output at $x = b$, then the graph of f changes from decreasing to increasing at $x = b$. Therefore, the graph of f' changes from being below the x-axis to being above the x-axis at $x = a$.

Example 2 *The diagram below shows the graph of the derivative function for a function f. Based on this graph, at which value of x does f have a relative maximum output? At which value of x does f have a relative minimum output? At which value of x does f have an inflection point?*



Graph of the derivative function for f

Solution. Notice that the graph of f' changes from being below the x-axis to being above the x-axis at the input value $x = -1$. For this reason, we know the graph of the function f changes from decreasing to increasing at this input value. Therefore, the function f will have a relative minimum output at $x = -1$. (We do not have enough information to determine what this output would be.)

On the other hand, notice that the graph of f' changes from being above the x-axis to being below the x-axis at the input value $x = 2$. For this reason, we know the graph of the function f changes from increasing to decreasing at this input value. Therefore, the function f will have a relative maximum output at $x = 2$.

We have already discussed the fact that the *derivative* function will have a turning point (relative maximum or relative minimum output) at an input value where the function f has an inflection point. Therefore, we see that the function f will have an inflection point at $x = 1/2$. The graph of the function f will change concavity at this input value.

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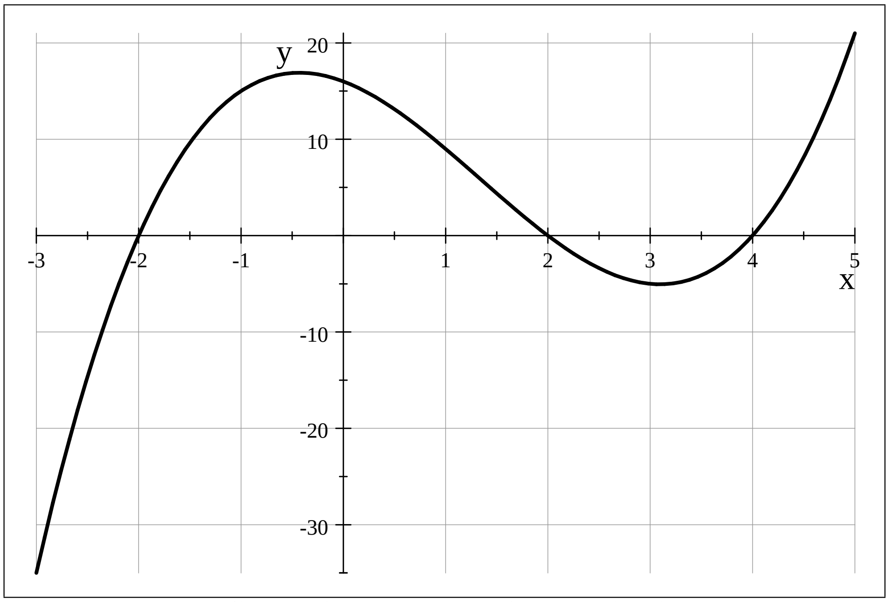
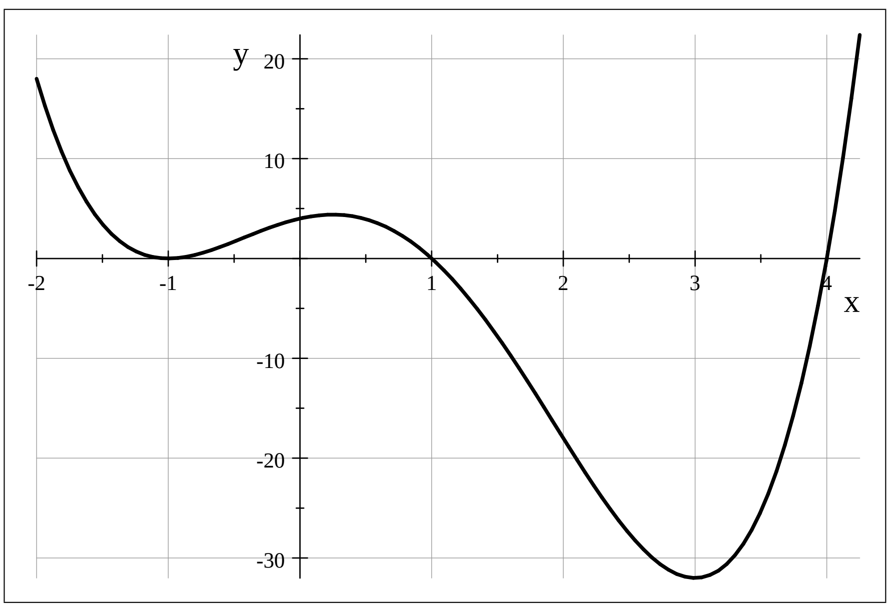
Problem 9. The graph of the derivative function for a function f is shown below. Use this graph to determine the input values where f has a relative maximum output, a relative minimum output, or an inflection point.



Graph of the derivative function for f

Problem 10. The graph of the derivative function for a function f is shown below. Use this graph to determine the input values where f has a relative maximum output, a relative minimum output, or an inflection point.



Graph of the derivative function for f