DERIVATIVE FORMULAS (PART 1)

In these notes, we will explore shortcuts for computing the derivative of certain classes and combinations of functions.

Problem 1. Consider the functions $f(x) = x^2$ and $g(x) = 5x^2$.

Part (a): How do the outputs of the functions f and g compare?

Part (b): Use the limit definition to compute the formula for the derivative function of f.

Part (c): Use the limit definition to compute the formula for the derivative function of g.

Part (d): How do the outputs of f' and g' compare?

Problem 2. Consider the functions $h(x) = x$ and $k(x) = \sqrt{3} \cdot x$.

Part (a): How do the outputs of the functions h and k compare?

Part (b): Use the limit definition to compute the formula for the derivative function of h.

Part (c): Use the limit definition to compute the formula for the derivative function of k.

Part (d): How do the outputs of h' and k' compare?

Problem 3. Consider the function $p(x) = x^2 + \sqrt{3} \cdot x$.

Part (a): How does the output of p compare to the outputs of f and k from Problems 1 and 2?

Part (b): Use the limit definition to compute the formula for the derivative function of p.

Part (c): How does the output of p' compare to the output of f' and k'?

Problems 1 - 3 illustrate two basic facts about the process of differentiation.

CONSTANT MULTIPLE RULE

- If f is any differentiable function and K is any fixed constant, then the derivative function for Kf is K times the derivative function for f. In symbols,

$$\underbrace{(K \cdot f)' = K \cdot f'}_{\text{Prime Notation}} \qquad \text{OR} \qquad \underbrace{\frac{d}{dx}\left[K \cdot f(x)\right] = K \cdot \frac{d}{dx}\left[f(x)\right]}_{\text{Differential Notation}}$$

SUM RULE

- If f and g are any differentiable functions, then the derivative function for $f+g$ is the sum of the derivative functions for f and g. In symbols,

$$(f+g)' = f' + g' \qquad \text{OR} \qquad \frac{d}{dx}\left[f(x)+g(x)\right] = \frac{d}{dx}\left[f(x)\right] + \frac{d}{dx}\left[g(x)\right]$$

Example 1 *Use the Sum and Constant Multiple Rules to compute the formula for the derivative function of* $g(a) = 5a^3 + 2a - \dfrac{3}{a}$.

Solution. The process of determining the formula for the derivative function of g is called *differentiating* the function g. This process involves applying general rules to break up the function into simpler and simpler pieces, until we reach specific functions whose derivatives we know.

$$\begin{aligned}
g'(a) &= \frac{d}{da}\left[5a^3 + 2a - \frac{3}{a}\right] \\
&= \frac{d}{da}\left[5a^3\right] + \frac{d}{da}\left[2a\right] + \frac{d}{da}\left[\frac{-3}{a}\right] && \text{Apply the Sum Rule} \\
&= 5\frac{d}{da}\left[a^3\right] + 2\frac{d}{da}\left[a\right] - 3\frac{d}{da}\left[\frac{1}{a}\right] && \text{Apply Constant Multiple Rule} \\
&= 5 \cdot \left(3a^2\right) + 2 \cdot (1) - 3 \cdot \left(-\frac{1}{a^2}\right) && \text{Apply specific derivative formulas} \\
&= 15a^2 + 2 + \frac{3}{a^2}
\end{aligned}$$

Problem 4. Using the previous example as a guide, differentiate the function $f(t) = 2t^2 - \dfrac{4}{t}$. Go through all of the steps used in the example.

It is fairly easy to see why the Sum and Constant Multiple Rules are consequences of the limit definition for the derivative. Suppose that f and g are differentiable functions at the input value $x = a$, and suppose that K is a fixed constant. Since f and g are differentiable at $x = a$, we know $f'(a)$ and $g'(a)$ both exist. In particular, at any input value $x = a$ we know

$$f'(a) = \lim_{h \longrightarrow 0} \frac{f(a+h) - f(a)}{h} \qquad \text{and} \qquad g'(a) = \lim_{h \longrightarrow 0} \frac{g(a+h) - g(a)}{h}$$

First, consider the function $p(x) = K \cdot f(x)$. If we set up the derivative limit for p, we find

$$p'(a) = \lim_{h \longrightarrow 0} \frac{p(a+h) - p(a)}{h} = \lim_{h \longrightarrow 0} \frac{K \cdot f(a+h) - K \cdot f(a)}{h} = \lim_{h \longrightarrow 0} K \cdot \left(\frac{f(a+h) - f(a)}{h} \right) = K \cdot f'(a)$$

Second, consider the function $s(x) = f(x) + g(x)$. If we set up the derivative limit for s, we find

$$\begin{aligned}
s'(a) &= \lim_{h \longrightarrow 0} \frac{s(a+h) - s(a)}{h} \\
&= \lim_{h \longrightarrow 0} \frac{[f(a+h) + g(a+h)] - [f(a) + g(a)]}{h} \\
&= \lim_{h \longrightarrow 0} \frac{[f(a+h) - f(a)] + [g(a+h) - g(a)]}{h} \\
&= \lim_{h \longrightarrow 0} \left[\frac{f(a+h) - f(a)}{h} + \frac{g(a+h) - g(a)}{h} \right] = f'(a) + g'(a)
\end{aligned}$$

HOMEWORK: Section 3.1 Page 180 Problems 7, 9, 21

Example 2 *Use the limit definition to determine the formula for the derivative function of $f(x) = \sqrt{x}$.*

Solution. To compute the formula for this derivative, we will use an algebra trick — if A and B are positive expressions, then

$$A - B = \left(\sqrt{A} - \sqrt{B} \right) \left(\sqrt{A} + \sqrt{B} \right)$$

With this in mind, observe that

$$\begin{aligned}
f'(x) = \frac{d}{dx}\left[\sqrt{x} \right] &= \lim_{h \longrightarrow 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \\
&= \lim_{h \longrightarrow 0} \left[\left(\frac{\sqrt{x+h} - \sqrt{x}}{h} \right) \left(\frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}} \right) \right] \\
&= \lim_{h \longrightarrow 0} \left[\frac{\left(\sqrt{x+h} - \sqrt{x} \right) \left(\sqrt{x+h} + \sqrt{x} \right)}{h \left(\sqrt{x+h} + \sqrt{x} \right)} \right] \\
&= \lim_{h \longrightarrow 0} \left[\frac{(x+h) - h}{h \left(\sqrt{x+h} + \sqrt{x} \right)} \right] \\
&= \lim_{h \longrightarrow 0} \frac{h}{h \left(\sqrt{x+h} + \sqrt{x} \right)} \\
&= \lim_{h \longrightarrow 0} \frac{1}{\sqrt{x+h} + \sqrt{x}} \\
&= \frac{1}{2\sqrt{x}}
\end{aligned}$$

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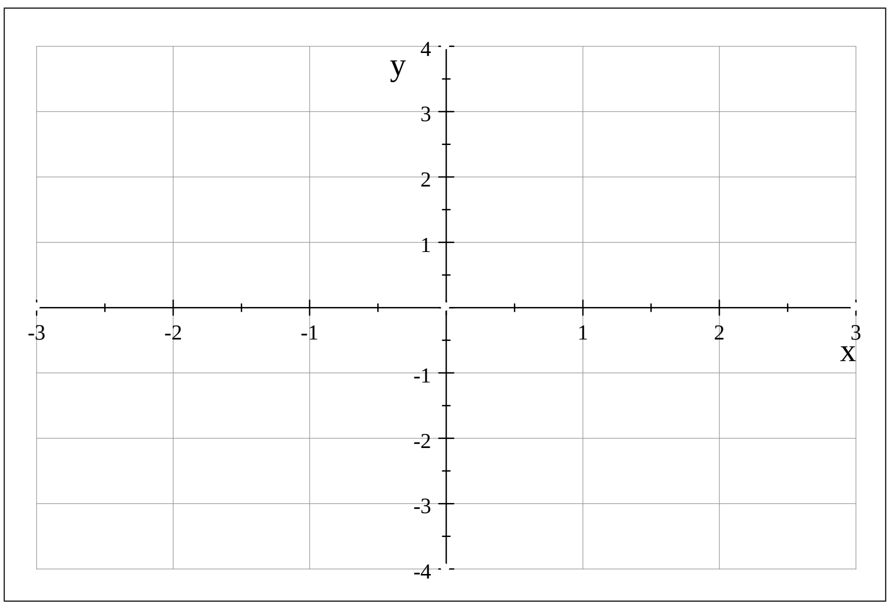
Problem 5. Consider the function $p(z) = 2z^3 + \dfrac{3}{z} - \sqrt{z}$.

Part (a): Use the Sum Rule and the Constant Multiple Rule to differentiate p. Go through all of the steps outlined in Example 1. .

Part (b): Construct the point-slope formula for the tangent line to p at the point $(4, p(4))$.

Problem 6. Now consider the *constant* function $y = f(x) = 2$.

Part (a): Draw the graph of this function on the grid provided.



Part (b): What is the slope of the tangent line to the graph of f at the point $(1, f(1))$? What is the slope of the tangent line to the graph of f at the point $(-2, f(-2))$?

Part (c): What do you think the formula for $f'(x)$ should be?

Part (d): Use the limit definition of the derivative to determine the formula for $f'(x)$.

The previous exercise provides motivation for the following observation.

$$\text{If } K \text{ is any fixed number, then } \frac{d}{dx}[K] = 0.$$

In other words, if $f(x) = K$ is a constant function, then the slope of the tangent line to the graph of f at a point will always be 0, no matter what point on the graph we consider.

Problem 7. Consider the function $q(m) = 4m^2 - 2m + 9$.

Part (a): Use the Sum Rule and the Constant Multiple Rule to determine the formula for q'.

Part (b): Use your formula from Part (a) to determine the m-value where the tangent line to q will have slope -10.

Part (c): Use your formula from Part (a) to determine the m-value where the tangent line to q will be horizontal.

HOMEWORK: Section 3.1 Pages 180-181, Problems 5, 8, 15, 17, 24, 43, 51, 55

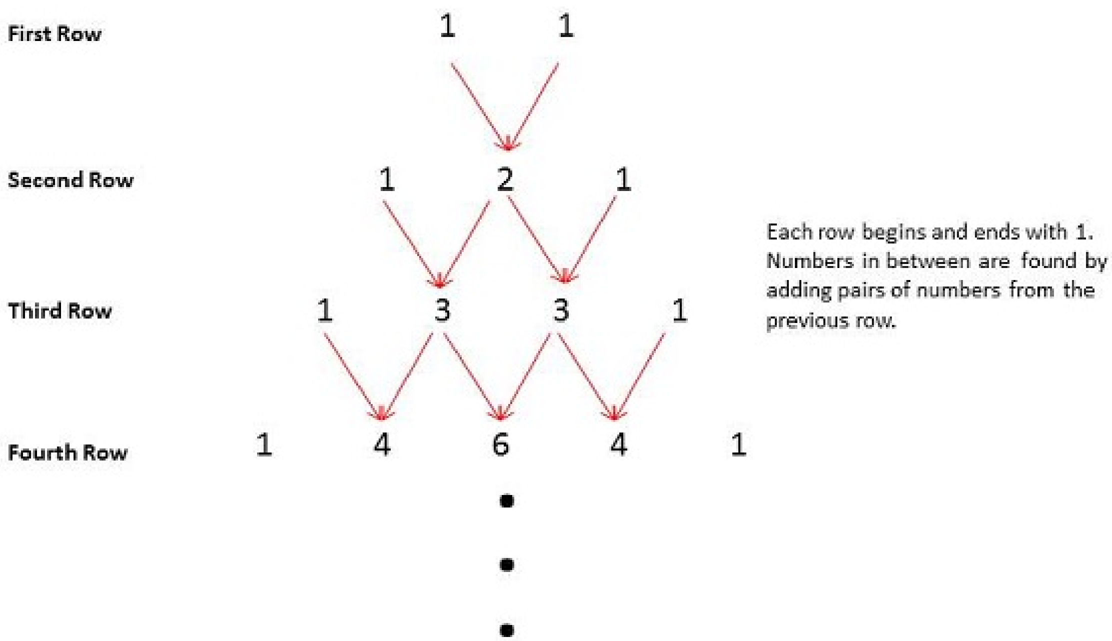
It is possible to construct the formula for the derivative of $f(x) = x^n$ for any positive integer power. However, doing so will require us to expand the binomial $(x-h)^n$, and this quickly becomes a daunting task. For example,

$$(x-h)^5 = (x-h)(x-h)(x-h)(x-h)(x-h)$$

and it is no small feat to perform the multiplication on the right hand side of this equation. Fortunately, there is a shortcut to expanding binomials raised to positive integer powers. There exist constants A_1 throuth A_{n-1} such that

$$(x+h)^n = x^n + A_1 x^{n-1} h + A_2 x^{n-2} h^2 + A_3 x^{n-3} h^3 + ... + A_{n-1} x h^{n-1} + h^n$$

Furthermore, these constants come from the nth row of *Pascal's Triangle.*



PASCAL'S TRIANGLE

For example, to expand $(x+h)^3$ we go to the third row of the triangle:

$$(x+h)^3 = x^3 + 3x^2 h + 3xh^2 + h^3$$

Example 3 *Use the limit definition to determine the formula for the derivative function of $f(x) = x^4$.*

Solution. Observe that

$$\begin{aligned} f'(x) = \frac{d}{dx}\left[x^3\right] &= \lim_{h \longrightarrow 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \longrightarrow 0} \frac{(x+h)^4 - x^4}{h} \\ &= \lim_{h \longrightarrow 0} \frac{\left[x^4 + 4x^3 h + 6x^2 h^2 + 4xh^3 + h^4\right] - x^4}{h} \\ &= \lim_{h \longrightarrow 0} \frac{4x^3 h + 6x^2 h^2 + 4xh^3 + h^4}{h} \\ &= \lim_{h \longrightarrow 0} \frac{h\left(4x^3 + 6x^2 h + 4xh^2 + h^3\right)}{h} \\ &= \lim_{h \longrightarrow 0} \left(4x^3 + 6x^2 h + 4xh^2 + h^3\right) \\ &= 4x^3 \end{aligned}$$

Problem 8. Construct the fifth row of Pascal's Triangle and use it to expand $(x+h)^5$.

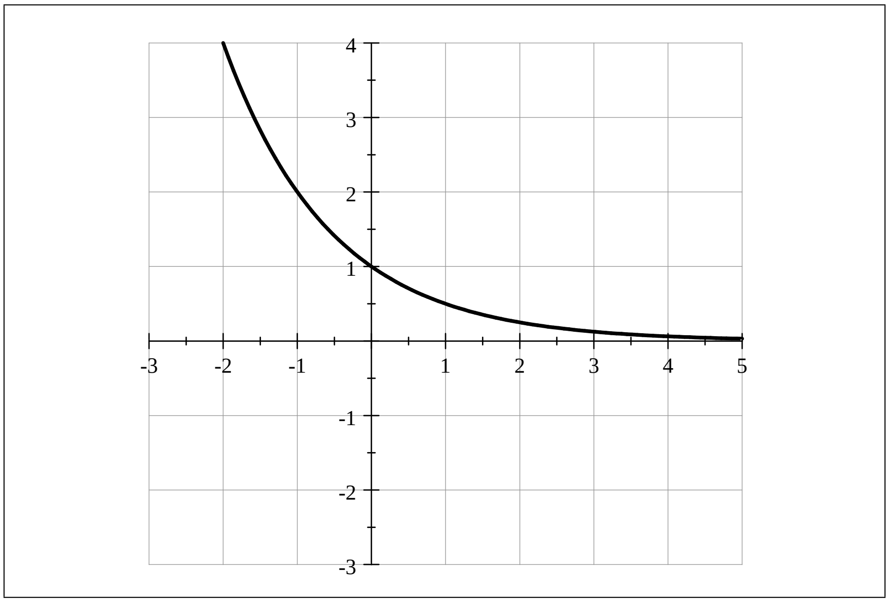
Problem 9. Use the limit definition to compute the formula for the derivative function of $f(x) = x^5$. You will need Problem 8.

There is a clear pattern emerging that we can summarize in the following rule:

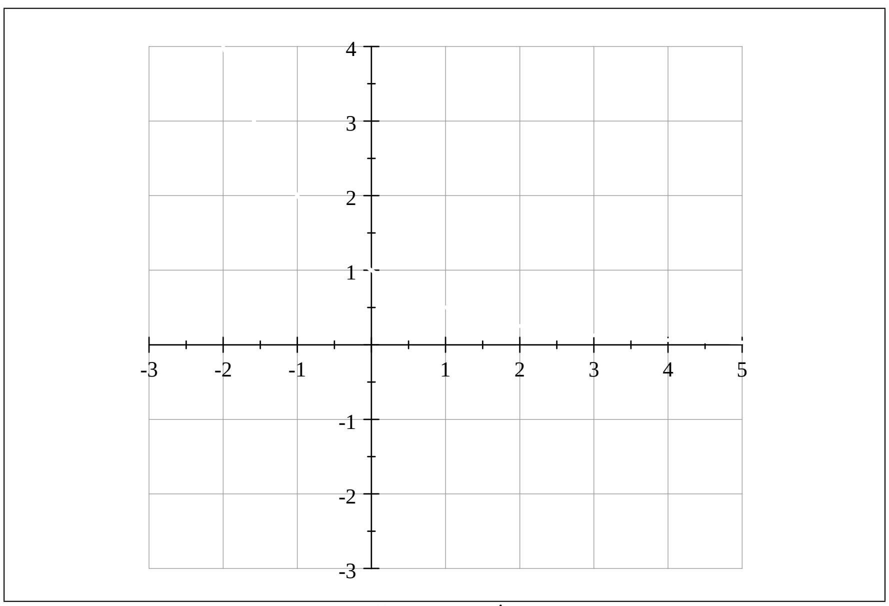
POWER RULE: If n is a positive integer, then $\dfrac{d}{dx}[x^n] = nx^{n-1}$.

Problem 10. Use the Power Rule along with the Sum and Constant Multiple Rules to compute the formula for the derivative function of $f(x) = 4x^6 - 3x^3 + \sqrt{x}$.

Problem 11. Consider the graph of a function f shown below. On the grid provided, sketch the graph of the derivative function for f. Be attentive to intervals where the graph of f is increasing or decreasing and the effect this has on the derivative of f.



Graph of f



Graph of f'

The function f is the base-2 exponential function $y = f(x) = 2^x$. Your sketch of the function f' for this function probably looks a little like the *reflection* of the graph for f around the x-axis. Is this a feature for exponential functions in general? To answer this question, we will have to look at base-B exponential functions in general. A bit of review is probably in order.

Consider the base-B exponential function $y = f(x) = B^x$, where B is any fixed positive number. This function represents a relationship of growth (or decay) by a fixed percentage. (The fixed percentage change is $r = B - 1$.) For example, the function $y = f(x) = 0.967^x$ represents a relationship between x and y in which a unit increase in the value of x always produces an percentage increase of -0.033 (that is, -3.3%) in the value of y. (We interpret the negative percentage to mean there is a *decrease* in the values of y of 3.3% for every unit change in the value of x.)

It is possible to determine a formula for the derivative of the base-B exponential function. To do this, we will need to use the laws of exponents and a bit of patience. In particular, we will need the fact that

$$B^{m+n} = B^m \cdot B^n$$

With this in mind, observe that

$$\begin{aligned} f'(x) = \frac{d}{dx}\left[B^x\right] &= \lim_{h \longrightarrow 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \longrightarrow 0} \frac{B^{x+h} - B^x}{h} \\ &= \lim_{h \longrightarrow 0} \frac{B^x B^h - B^x}{h} \\ &= \lim_{h \longrightarrow 0} B^x \cdot \left(\frac{B^h - 1}{h}\right) \end{aligned}$$

From the last expression, we can see that the derivative formula for $f(x) = B^x$ really depends on what happens to the function

$$L(h) = \frac{B^h - 1}{h}$$

as the values of h approach 0. Once again, the function L has a discontinuity at $h = 0$. Unfortunately, it is obvious what the behavior of L is near $h = 0$. This is because, in order to examine this behavior, we will need to check the behavior for different values of the base B. Suppose we let $B = 2$. Here is a table showing the output of the function L for values of h very close to 0.

Value of h	-0.0003	-0.0002	-0.0001	0.0000	0.0001	0.0002	0.0003
Value of L(h)	0.69308	0.69310	0.69312	ERROR	0.69317	0.69320	0.69322

From this table, we can conclude that

$$\lim_{h \longrightarrow 0} \frac{2^h - 1}{h} \approx 0.6931$$

This tells us that

$$\frac{d}{dx}\left[2^x\right] \approx 0.6931 \cdot 2^x$$

Problem 12. Let's examine the behavior of the function L near $h = 0$ for some other values of B.

Part (a): Complete the following tables using the table feature on your graphing calculator.

$$L(h) = \frac{0.5^h - 1}{h}$$

Value of h	-0.0003	-0.0002	-0.0001	0.0000	0.0001	0.0002	0.0003
Value of L(h)							

$$L(h) = \frac{4^h - 1}{h}$$

Value of h	-0.0003	-0.0002	-0.0001	0.0000	0.0001	0.0002	0.0003
Value of L(h)							

$$L(h) = \frac{0.25^h - 1}{h}$$

Value of h	-0.0003	-0.0002	-0.0001	0.0000	0.0001	0.0002	0.0003
Value of L(h)							

Part (b): Based on the results of your tables, complete the following.

$$(1) \quad \lim_{h \longrightarrow 0} \frac{0.5^h - 1}{h} \approx \qquad \frac{d}{dx}[0.5^x] \approx$$

$$(1) \quad \lim_{h \longrightarrow 0} \frac{4^h - 1}{h} \approx \qquad \frac{d}{dx}[4^x] \approx$$

$$(1) \quad \lim_{h \longrightarrow 0} \frac{0.25^h - 1}{h} \approx \qquad \frac{d}{dx}[0.25^x] \approx$$

At the moment, it is probably not obvious what the general pattern is. However, there are some patterns that seem to be emerging. For example, we know that the value of

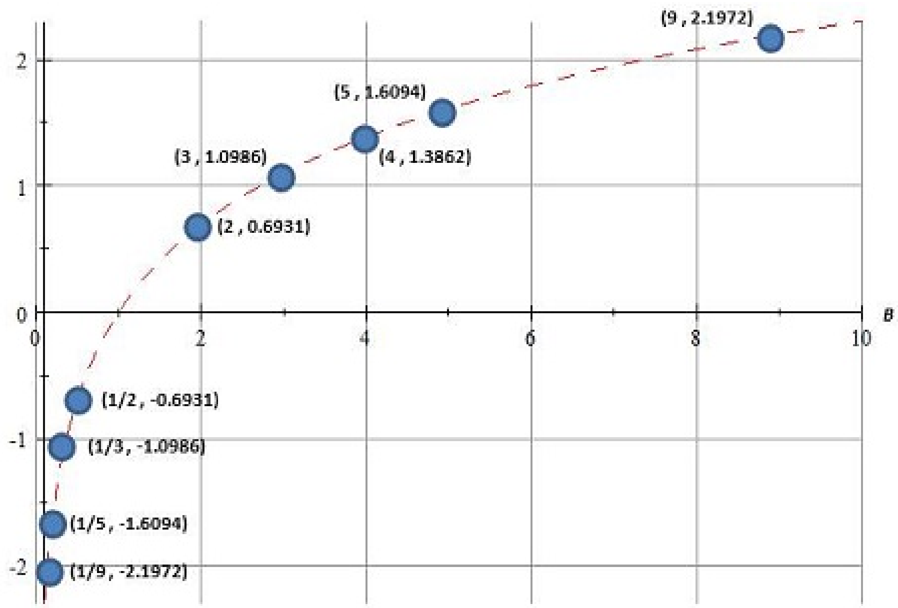
$$\lim_{h \longrightarrow 0} \frac{4^h - 1}{h} \text{ is approximately twice the value of } \lim_{h \longrightarrow 0} \frac{2^h - 1}{h}$$

$$\lim_{h \longrightarrow 0} \frac{0.5^h - 1}{h} \text{ and } \lim_{h \longrightarrow 0} \frac{0.25^h - 1}{h} \text{ are the negatives of the value of } \lim_{h \longrightarrow 0} \frac{2^h - 1}{h} \text{ and } \lim_{h \longrightarrow 0} \frac{4^h - 1}{h}$$

This kind of pattern continues. Here are the approximate limiting values of $L(h)$ near $h = 0$ for other choices of B.

Value of B	1/9	1/5	1/3	3	5	9
$\lim_{h\to 0} \frac{B^h - 1}{h}$	−2.1972	−1.6094	−1.0986	1.0986	1.6094	2.1972

Now, let's plot the ordered pairs $\left(B, \lim_{h\longrightarrow 0} \dfrac{B^h - 1}{h}\right)$.



When we plot the ordered pairs, we can see that they all lie on the graph of the function $y = \ln(B)$. We will therefore conclude that

$$\lim_{h\longrightarrow 0} \frac{B^h - 1}{h} = \ln(B) \qquad \text{and} \qquad \frac{d}{dx}\left[B^x\right] = \lim_{h\longrightarrow 0}\left(B^x \cdot \frac{B^h - 1}{h}\right) = B^x \ln(B)$$

Problem 13. What is the slope-intercept formula for the tangent line to the function $f(x) = 6^x$ at the point $(-2, f(-2))$?

Problem 14. What is the slope-intercept formula for the tangent line to the function $f(x) = 0.98^x$ at the point $(3, f(3))$?

Problem 15. What is the derivative function for $f(x) = e^x$, where e is Euler's constant?

Problem 16. Use the Sum and Constant Multiple Rules to differentiate the function $h(s) = 4e^s - s^8 + \dfrac{2}{s}$. Go through all of the steps in Example 1.

HOMEWORK: Section 3.1 Pages 180 - 181, Problems 3, 4, 32, 34, 56 (Use the laws of exponents on Problem 32 before differentiating.)