DERIVATIVE FORMULAS (PART II)

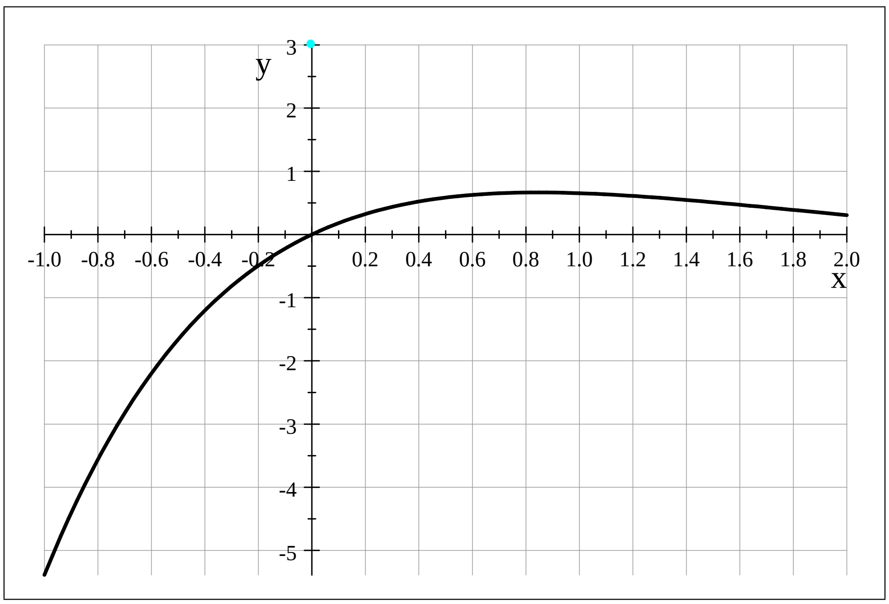
Let's start by using the graphing calculator to sketch the graph of the function $f(x) = x^2 \left(\frac{1}{2}\right)^x$.

Using Your Calculator to Sketch the Derivative Graph of a Function

1. Enter your function formula as Y_1 on your calculator. ($Y_1 = x\hat{\ }2 * 0.5\hat{\ }x$)
2. Turn off the function Y_1.
3. Select function Y_2.
4. Select `MATH` Option 8 `nDeriv(`
5. Enter Y_1 by selecting `VARS` followed by `Y-VARS` followed by `FUNCTION` (Option 1) followed by selecting function Y_1.
6. Enter a comma, followed by `X,` followed by another comma, followed by `X` and close the parentheses.

```
Y2 = nDeriv(Y1,X,X)
```

Set your viewing window for $-1 \le x \le 2$ and $-5 \le y \le 3$ and graph the function Y_2. You should see the following graph appear on your view screen.



Problem 1. Let $f(x) = x^2$ and let $g(x) = \left(\frac{1}{2}\right)^x$. Construct the formulas for

$$f'(x) = \frac{d}{dx}\left[x^2\right] \qquad \text{and} \qquad g'(x) = \frac{d}{dx}\left[\left(\frac{1}{2}\right)^x\right]$$

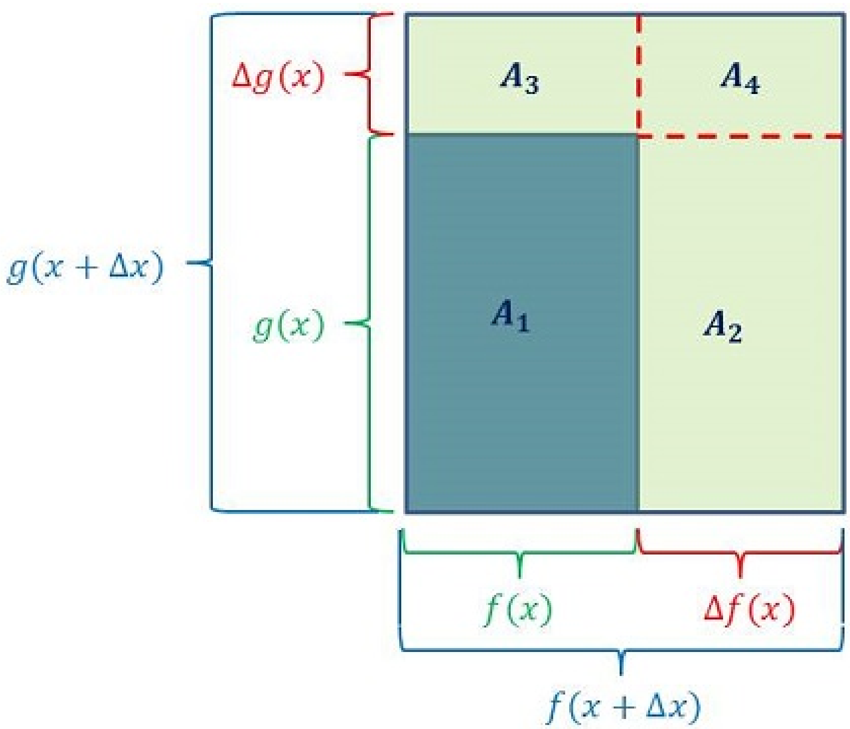
Problem 2. On your graphing calculator, let $Y_3 = f'(x)g'(x)$ and graph this function along with the derivative above. Are these graphs the same?

PRODUCT RULE

- If g and h are differentiable functions, then their product $f = g \cdot h$ is also differentiable, and

$$\underbrace{(g \cdot h)'(x) = (g \cdot h')(x) + (h \cdot g')(x)}_{\text{Prime Notation}} \qquad \text{OR} \qquad \underbrace{\frac{d}{dx}[g(x) \cdot h(x)] = g(x) \cdot \frac{d}{dx}[h(x)] + h(x) \cdot \frac{d}{dx}[g(x)]}_{\text{Differential Notation}}$$

We can motivate this formula in the following way. Suppose that we have a rectangle whose side lengths can vary with time. Let the width of the rectangle in feet at time x seconds be given by $w = f(x)$ and let the height of the rectangle in feet at time x seconds be given by $h = g(x)$. At x seconds, the area of the rectangle is given by $A(x) = f(x)g(x)$. Now, suppose that we let the time increase from x seconds to $x + \triangle x$ seconds. What will the area of the new rectangle be?



Based on the diagram above, the area of the new rectangle will be

$$\begin{aligned} A(x + \triangle x) &= f(x + \triangle x)g(x + \triangle x) \\ &= A_1 + A_2 + A_3 + A_4 \\ &= f(x)g(x) + g(x)\triangle f(x) + f(x)\triangle g(x) + \triangle f(x)\triangle g(x) \end{aligned}$$

Now, the *average* rate of change for the area of the rectangle on the time interval $[x, x + \triangle x]$ will be

$$\begin{aligned} \frac{A(x + \triangle x) - A(x)}{(x + \triangle x) - x} &= \frac{[f(x)g(x) + g(x)\triangle f(x) + f(x)\triangle g(x) + \triangle f(x)\triangle g(x)] - f(x)g(x)}{\triangle x} \\ &= \frac{g(x)\triangle f(x) + f(x)\triangle g(x) + \triangle f(x)\triangle g(x)}{\triangle x} \\ &= g(x)\frac{\triangle f(x)}{\triangle x} + f(x)\frac{\triangle g(x)}{\triangle x} + \frac{\triangle f(x)\triangle g(x)}{\triangle x} \end{aligned}$$

If we make the assumption that the functions f and g are *differentiable*, then we can determine a formula for the instantaneous rate of change for the area function. Observe

$$
\begin{aligned}
A'(x) &= \lim_{\triangle x \longrightarrow 0} \left[g(x)\frac{\triangle f(x)}{\triangle x} + f(x)\frac{\triangle g(x)}{\triangle x} + \frac{\triangle f(x) \triangle g(x)}{\triangle x} \right] \\
&= g(x) \lim_{\triangle x \longrightarrow 0} \frac{\triangle f(x)}{\triangle x} + f(x) \lim_{\triangle x \longrightarrow 0} \frac{\triangle g(x)}{\triangle x} + \lim_{\triangle x \longrightarrow 0} \triangle f(x) \cdot \lim_{\triangle x \longrightarrow 0} \frac{\triangle g(x)}{\triangle x} \\
&= g(x)f'(x) + f(x)g'(x) + g'(x) \lim_{\triangle x \longrightarrow 0} \triangle f(x) \\
&= g(x)f'(x) + f(x)g'(x) + g'(x) \lim_{\triangle x \longrightarrow 0} [f(x + \triangle x) - f(x)] \\
&= g(x)f'(x) + f(x)g'(x) + g'(x)\,[f(x) - f(x)] \\
&= g(x)f'(x) + f(x)g'(x)
\end{aligned}
$$

Example 1 *Use the derivative rules and formulas we have developed to differentiate the function* $f(r) = 3\sqrt{r} - 4r^3 \cdot 2^r$.

Solution. We use the derivative rules to break up the formula into derivatives of smaller and smaller parts until we reach formulas whose derivatives we know.

$$
\begin{aligned}
\frac{d}{dr}\left[3\sqrt{r} - 4r^3 \cdot 2^r\right] &= \frac{d}{dr}\left[3\sqrt{r}\right] + \frac{d}{dr}\left[-4r^3 \cdot 2^r\right] && \text{Apply Sum Rule} \\
&= 3\frac{d}{dr}\left[\sqrt{r}\right] - 4\frac{d}{dr}\left[r^3 \cdot 2^r\right] && \text{Apply Constant Multiple Rule} \\
&= 3\frac{d}{dr}\left[\sqrt{r}\right] - 4\left(r^3 \cdot \frac{d}{dr}\left[2^r\right] + 2^r \cdot \frac{d}{dr}\left[r^3\right]\right) && \text{Apply Product Rule} \\
&= 3\left(\frac{1}{2\sqrt{r}}\right) - 4\left(r^3 \cdot 2^r \ln(2) + 2^r \cdot \left(3r^2\right)\right) && \text{Apply Specific Derivative Formulas} \\
&= \frac{3}{2\sqrt{r}} - 4 \cdot 2^r \left(r^3 \ln(2) + 3r^2\right)
\end{aligned}
$$

Problem 3. Follow the procedure in Example 1 to differentiate the function $g(y) = y^2 - \sqrt{y} \cdot y^5$.

Problem 4. Follow the procedure in Example 1 to differentiate the function $h(t) = \dfrac{3}{t} - 5t^3 \cdot e^t$.

Problem 5. Use the fact that $x^{3/2} = x \cdot \sqrt{x}$ and the Product Rule to differentiate the function $f(x) = x^{3/2}$.

HOMEWORK: Section 3.2 Page 188 Problems 3, 4, 9, 12, 37, 43, 45

There is another rule that gives us a way to compute derivative formulas for quotients of differentiable functions. This rule is not as useful as the Product Rule, but it can be convenient to use at times.

QUOTIENT RULE

- If g and h are differentiable functions, then their quotient $f = \dfrac{g}{h}$ is also differentiable as long as f is defined, and

$$\underbrace{\left(\frac{g}{h}\right)' = \frac{h \cdot g' - g \cdot h'}{h^2}}_{\text{Prime Notation}} \qquad \text{OR} \qquad \underbrace{\frac{d}{dx}\left[\frac{g(x)}{h(x)}\right] = \left(\frac{1}{[h(x)]^2}\right)\left(h(x) \cdot \frac{d}{dx}[g(x)] - g(x) \cdot \frac{d}{dx}[h(x)]\right)}_{\text{Differential Notation}}$$

Example 2 *Use the Quotient Rule to determine the points where the tangent line to* $f(x) = \dfrac{x^2-1}{3x^2+1}$ *will be horizontal.*

Solution. We use the derivative rules to break up the formula into derivatives of smaller and smaller parts until we reach formulas whose derivatives we know. It helps to apply the Quotient Rule in two steps. First, let

$$g(x) = x^2 - 1 \quad \text{so that} \quad g'(x) = 2x$$
$$h(x) = 3x^2 + 1 \quad \text{so that} \quad h'(x) = 6x$$

$$\begin{aligned}
\frac{d}{dx}\left[\frac{x^2-1}{3x^2+1}\right] &= \frac{h \cdot g' - g \cdot h'}{h^2} \\
&= \frac{(3x^2+1)(2x) - (x^2-1)(3x)}{[3x^2+1]^2} \\
&= \frac{6x^3 + 2x - 3x^3 + 3x}{[3x^2+1]^2} \\
&= \frac{3x^3 + 5x}{[3x^2+1]^2}
\end{aligned}$$

The formula we have just derived will give us the slope of the tangent line to f at any input value $x = a$. Since we want to know the value of x when the tangent line is horizontal, we want to solve the equation

$$\begin{aligned} \frac{3x^3 + 5x}{\left[3x^2 + 1\right]^2} = 0 \quad &\Longrightarrow \quad 3x^3 + 5x = 0 \\ &\Longrightarrow \quad x(3x^2 + 5) = 0 \\ &\Longrightarrow \quad x = 0 \qquad \text{or} \qquad 3x^2 + 5 = 0 \end{aligned}$$

The equation $3x^2 + 5 = 0$ has no real number solution, so the tangent line to f will be horizontal only when $x = 0$. The point of tangency will be

$$(0, f(0)) = (0, -1)$$

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Problem 6. Use the Quotient Rule to determine the input values where the tangent line to $f(y) = \dfrac{2y}{3y^2 - 1}$ will have slope -8.

We can use the Quotient Rule to extend the power rule to include negative integer powers. Suppose that n is a positive integer. Observe that

$$\frac{d}{dx}\left[x^{-n}\right] = \frac{d}{dx}\left[\frac{1}{x^n}\right]$$

Let $g(x) = 1$ and let $h(x) = x^n$. The Power Rule tells us that $h'(x) = nx^{n-1}$. Since g is a constant function, we also know that $g'(x) = 0$. With this in mind, observe

$$\begin{aligned}
\frac{d}{dx}\left[\frac{1}{x^n}\right] &= \frac{h \cdot g' - g \cdot h'}{h^2} \\
&= \frac{x^n \cdot 0 - 1 \cdot nx^{n-1}}{\left[x^n\right]^2} \\
&= -\frac{nx^{n-1}}{x^{2n}} \\
&= -n \cdot x^{(n-1)-2n} \\
&= -n \cdot x^{-n-1} \\
&= -\frac{n}{x^{n+1}}
\end{aligned}$$

GENERAL POWER RULE

- If n is any nonzero integer, then $\frac{d}{dx}\left[x^n\right] = nx^{n-1}$.

Problem 7. Differentiate the function $f(x) = 5x^{-7} + 3x^3 + 4$.

HOMEWORK: Section 3.2 Page 180, Problems 5, 6, 7, 13, 14, 15, 19, 23, 31, 35