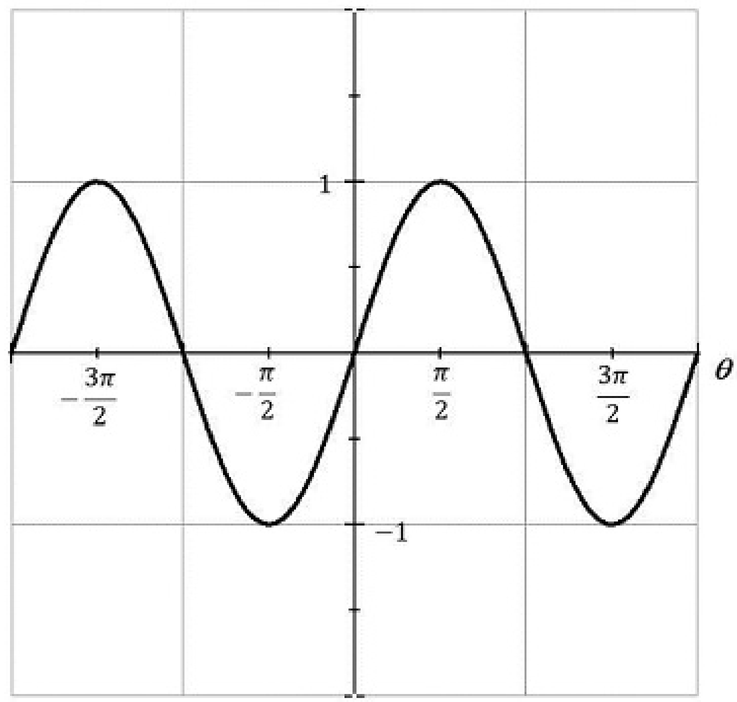
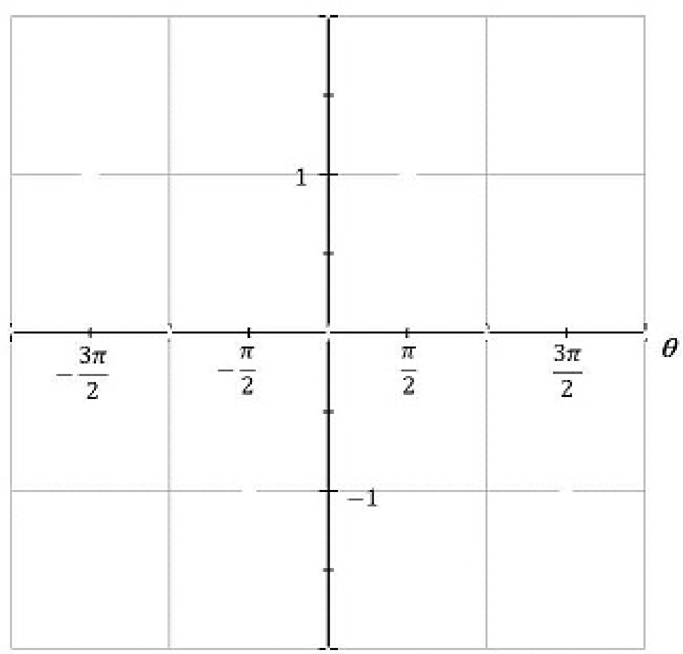
DERIVATIVES OF THE TRIGONOMETRIC FUNCTIONS

In these notes, we will explore shortcuts for computing the derivative of the trigonometric functions.

Problem 1. Consider the graph of a function $f(\theta) = \sin(\theta)$ shown below. On the grid provided, sketch the graph of the derivative function for f. Be attentive to intervals where the graph of f is increasing or decreasing, and to the turning and inflection points for f.

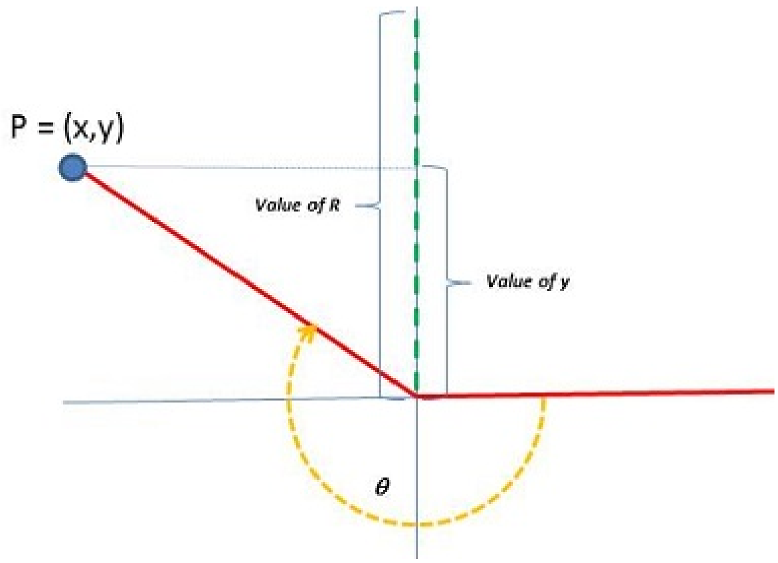




The function f in Problem 1 is the sine function $p = f(\theta) = \sin(\theta)$, and the graph you sketched represents the derivative function for the sine function $p = f'(\theta)$. If you compare the graph of f' with the graph of f, you can see that f' *looks a little like the cosine function.* Our next goal will be to offer some proof that the derivative of the sine function really is the cosine function. A little review is probably in order first.

Let $P = (x, y)$ be a point other than the vertex on the terminal side of an angle in standard position, and suppose θ is the radian measure of this angle. Let R be the distance from the origin to the point P. Recall that the function $p = f(\theta) = \sin(\theta)$ gives the percentage of R above or below the x-axis represented by y. In other words,

$$f(\theta) = \sin(\theta) = \frac{y}{R}$$



In similar fashion, the function $p = g(\theta) = \cos(\theta)$ gives the percentage of R right or left of the y-axis represented by x. In other words,

$$g(\theta) = \cos(\theta) = \frac{x}{R}$$

The sine and cosine functions are *transcendental* functions of the angle measure θ, because the value of θ does not appear explicitly in the formulas for computing their output.

We have assumed the length of the line segment connecting the point $P = (x, y)$ to the origin is R. The distance formula also tells us that

$$\begin{aligned}
R = \sqrt{(x-0)^2 + (y-0)^2} \quad &\Longrightarrow \quad R = \sqrt{x^2 + y^2} \\
&\Longrightarrow \quad R^2 = x^2 + y^2 \\
&\Longrightarrow \quad 1 = \frac{x^2 + y^2}{R^2} \\
&\Longrightarrow \quad 1 = \left(\frac{x}{R}\right)^2 + \left(\frac{y}{R}\right)^2 \\
&\Longrightarrow \quad 1 = [\cos(\theta)]^2 + [\sin(\theta)]^2
\end{aligned}$$

The *Pythagorean Identity* $1 = \cos^2(\theta) + \sin^2(\theta)$ is probably the most important relationship between the sine and cosine functions for Calculus I. Here are the other important identities that you will encounter frequently:

$$\text{Quotient Identity:} \quad \tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)}$$

$$\text{Reciprocal Identities:} \quad \sec(\theta) = \frac{1}{\cos(\theta)} \qquad \csc(\theta) = \frac{1}{\sin(\theta)} \qquad \cot(\theta) = \frac{1}{\tan(\theta)}$$

If we want to develop an explicit formula for the derivative of the sine function, we will have to resort to the limit definition of the derivative (because this is the only way we can determine such formulas). We can take advantage of a trigonometric identity to help us determine a formula for the derivative function of the sine function. The so-called *sum formula* for the sine function tells us

$$\sin(\theta + h) = \sin(\theta)\cos(h) + \cos(\theta)\sin(h)$$

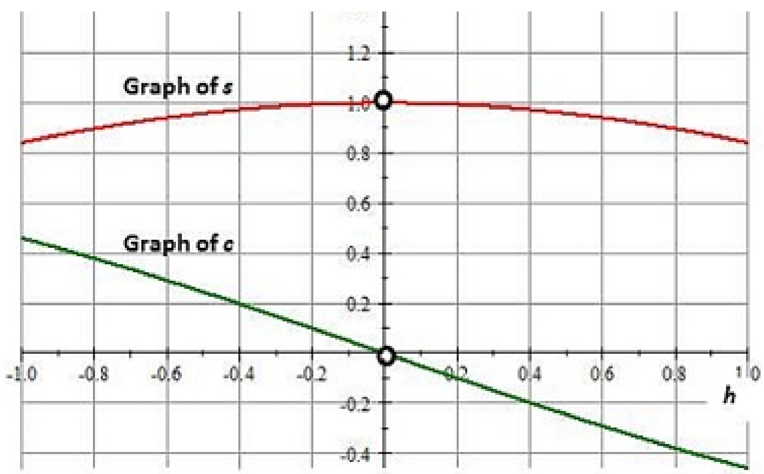
where θ and h are the radian measures of any two angles. (This formula is developed in most trigonometry books.) With this formula in mind, observe that

$$\begin{aligned} f'(\theta) = \frac{d}{d\theta}[\sin(\theta)] &= \lim_{h \longrightarrow 0} \frac{\sin(\theta+h) - \sin(\theta)}{h} \\ &= \lim_{h \longrightarrow 0} \frac{\sin(\theta)\cos(h) + \cos(\theta)\sin(h) - \sin(\theta)}{h} \\ &= \lim_{h \longrightarrow 0} \frac{\cos(\theta)\sin(h) - \sin(\theta)[\cos(h) - 1]}{h} \\ &= \lim_{h \longrightarrow 0} \left[\cos(\theta) \cdot \frac{\sin(h)}{h} - \sin(\theta) \cdot \frac{\cos(h) - 1}{h}\right] \end{aligned}$$

The last equation tells us that the formula for the derivative of the sine function really depends on what happens to the functions

$$s(h) = \frac{\sin(h)}{h} \qquad\qquad c(h) = \frac{\cos(h) - 1}{h}$$

as the input variable h approaches 0. Both of these functions have a discontinuity at $h = 0$; however, if we look at the graphs for both functions, we see that this discontinuity is removable.



Looking at these graphs, we can see that

$$f'(\theta) = \frac{d}{d\theta}[\sin(\theta)] = \cos(\theta) \cdot 1 - \sin(\theta) \cdot 0 = \cos(\theta)$$

We can use a similar approach to determine the formula for the derivative of the cosine function $g(\theta) = \cos(\theta)$. The *sum formula* for the cosine function tells us

$$\cos(\theta + h) = \cos(\theta)\cos(h) - \sin(\theta)\sin(h)$$

where θ and h are the radian measures of any two angles. (This formula is developed in most trigonometry books.) With this formula in mind, observe that

$$\begin{aligned}
g'(\theta) = \frac{d}{d\theta}[\cos(\theta)] &= \lim_{h \longrightarrow 0} \frac{\cos(\theta + h) - \cos(\theta)}{h} \\
&= \lim_{h \longrightarrow 0} \frac{\cos(\theta)\cos(h) - \sin(\theta)\sin(h) - \cos(\theta)}{h} \\
&= \lim_{h \longrightarrow 0} \frac{\cos(\theta)[\cos(h) - 1] - \sin(\theta)\sin(h)}{h} \\
&= \lim_{h \longrightarrow 0} \left[\cos(\theta) \cdot \frac{\cos(h) - 1}{h} - \sin(\theta) \cdot \frac{\sin(h)}{h}\right] \\
&= \cos(\theta) \cdot 0 - \sin(\theta) \cdot 1 \\
&= -\sin(\theta)
\end{aligned}$$

Example 1 *What is the slope-intercept formula for the tangent line to the function* $f(\theta) = \sin(\theta)$ *at the point* $(1.5, f(1.5))$*?*

Solution. The point-slope formula for the tangent line will be

$$y - f(1.5) = f'(1.5)[\theta - 1.5]$$

Now, we know that $f(1.5) = \sin(1.5) \approx 0.9975$. We also know that $f'(1.5) = \cos(1.5) \approx 0.0707$. Therefore, we know the point-slope formula for the tangent line will be

$$y - 0.9975 \approx 0.0707[\theta - 1.5]$$

Putting this formula into slope-intercept form, we have

$$y \approx 0.0707\theta + 0.89145$$

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Problem 2. What is the slope-intercept formula for the tangent line to the function $g(\theta) = \cos(\theta)$ at the point $(2.7, g(2.7))$?

Problem 3. Differentiate the function $f(t) = \sin(t)\cos(t)$.

Example 2 *Differentiate the function* $f(z) = \dfrac{\cos(z)}{1+\sin(z)}$.

Solution. Observe that we have a quotient of two functions. Let $g(z) = \cos(z)$ and let $h(z) = 1 + \sin(z)$. We know that $g'(z) = -\sin(z)$ and $h'(z) = \cos(z)$. With this in mind,

$$\begin{aligned}
f'(z) &= \frac{d}{dz}\left[\frac{\cos(z)}{1+\sin(z)}\right] \\
&= \frac{h(z)g'(z) - g(z)h'(z)}{h^2(z)} && \text{Apply Quotient Rule} \\
&= \frac{(1+\sin(z))\left[-\sin(z)\right] - \cos(z)\left[\cos(z)\right]}{(1+\sin(z))^2} \\
&= \frac{-\sin(z) - \sin^2(z) - \cos^2(z)}{(1+\sin(z))^2} \\
&= \frac{-\sin(z) - \left[\sin^2(z) + \cos^2(z)\right]}{(1+\sin(z))^2} \\
&= \frac{-\sin(z) - 1}{(1+\sin(z))^2} && \text{Apply the Pythagorean Identity to simplify}
\end{aligned}$$

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Problem 4. Use the fact that $\tan(\theta) = \dfrac{\sin(\theta)}{\cos(\theta)}$ to differentiate the function $g(\theta) = \tan(\theta)$. Use the Pythagorean and reciprocal identities to simplify your answer as much as you can.

Problem 5. Use the fact that $\sec(\theta) = \dfrac{1}{\cos(\theta)}$ to differentiate the function $h(\theta) = \sec(\theta)$. Use the quotient and reciprocal identities to simplify your answer as much as you can.