THE CHAIN RULE

In these notes, we will introduce and explore one of the most powerful general derivative rules. Let's begin with an example.

The temperature (in degrees Fahrenheit) of the outside of a 747 jet depends on the altitude of the plane. Let $T = f(u)$ be the function that gives the temperature in degrees Fahrenheit for the plane in terms of the altitude in feet of the plane above sea level. The altitude of the plane depends on the number of seconds since it left the airport. Let $u = g(x)$ be the function that gives the altitude in feet above sea level of the plane x seconds after takeoff.

Problem: How can we determine the instantaneous rate of change in the temperature of the outside of the plane with respect to the number of seconds since it left the airport?

Note that the instantaneous rate of change for the temperature with respect to the number of seconds since the plane left the airport is represented by the symbol

$$\frac{df}{dx} \quad \text{(Units are degrees Fahrenheit per second since departure)}$$

However, the function f is not defined directly in terms of the time variable x. The function f is defined directly in terms of the altitude variable u. Now, think about the units for the rates of change

$$\frac{df}{du} \quad \text{(Units are degrees Fahrenheit per foot of elevation)}$$

$$\frac{du}{dx} \quad \text{(Units are feet of elevation per second since departure)}$$

The units on these rates of change suggest that

$$\frac{df}{dx} = \frac{df}{du} \cdot \frac{du}{dx}$$

Let's check to see if this rule works. Suppose we know that

$$f(u) = \frac{u}{1+u} \qquad \text{and} \qquad u(x) = 100x$$

First, note that, under this assumption, we know

$$f(x) = f(u(x)) = \frac{100x}{1+100x}$$

Since we now have a function that gives the temperature directly in terms of the number of seconds since takeoff, we can compute $\frac{df}{dx}$ directly. Observe

$$\begin{aligned} f'(x) &= \frac{d}{dx}\left[\frac{100x}{1+100x}\right] \\ &= \frac{1}{(1+100x)^2}\left[(1+100x)\cdot\frac{d}{dx}[100x] - 100x\cdot\frac{d}{dx}[1+100x]\right] \\ &= \frac{1}{(1+100x)^2}\left[(1+100x)(100) - 100x(100)\right] \\ &= \frac{100}{(1+100x)^2} \end{aligned}$$

On the other hand, we also know

$$\frac{df}{du} = \frac{d}{du}\left[\frac{u}{1+u}\right] = \frac{1}{(1+u)^2} \qquad \text{and} \qquad \frac{du}{dx} = \frac{d}{dx}[100x] = 100$$

$$\begin{aligned}
\frac{df}{du} \cdot \frac{du}{dx} &= \left[\frac{1}{(1+u)^2}\right] \cdot (100) \\
&= \frac{100}{(1+u)^2} \\
&= \frac{100}{(1+100x)^2} \qquad \text{(Using the fact that } u = 100x\text{)}
\end{aligned}$$

The previous discussion motivates the last of the general derivative rules.

CHAIN RULE: If f is a differentiable function of u and u is a differentiable function of x, then

$$\frac{df}{dx} = \frac{df}{du} \cdot \frac{du}{dx}$$

Example 1 *Use the Chain Rule to differentiate the composite function $g(x) = \sin^2(x)$.*

Solution. First, observe that $g(x) = [\sin(x)]^2$. Let $u(x) = \sin(x)$ and $f(u) = u^2$ so that $g(x) = f(u(x))$. The Chain Rule tells us

$$\begin{aligned}
\frac{d}{dx}\left[\sin^2(x)\right] = \frac{df}{dx} &= \frac{df}{du} \cdot \frac{du}{dx} \\
&= \frac{d}{du}\left[u^2\right] \cdot \frac{d}{dx}[\sin(x)] \\
&= 2u \cdot \cos(x) \\
&= 2\sin(x)\cos(x)
\end{aligned}$$

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Problem 1. Use the Chain Rule to differentiate the function $g(x) = \cos^3(x)$.

Problem 2. Use the Chain Rule to differentiate the function $g(x) = \tan(x^3)$.

Problem 3. Use the Chain Rule to differentiate the function $g(x) = (x + \sin(x))^{-2}$.

HOMEWORK: Section 3.4 Page 204 Problems 7, 11, 12, 13, 14, 16

Example 2 *Use the Chain Rule to differentiate the function* $f(x) = x^{1/5}$.

Solution. To keep the variables cleaner, let's drop the function notation for a moment and rewrite the relationship as $y = x^{1/5}$. Now, differentiating both sides of this equation with respect to x gives us

$$y = x^{1/5} \Longrightarrow \frac{dy}{dx} = \frac{d}{dx}\left[x^{1/5}\right]$$

At the moment, we cannot take this derivative process any further, because the derivative on the right is not an exact match for any of our special formulas. Let's look at the relationhsip another way.

If we let $y = x^{1/5}$ then we also have $y^5 = x$.

Now, if we differentiate both sides of the relationship with respect to x we have

$$y^5 = x \Longrightarrow \frac{d}{dx}\left[y^5\right] = \frac{d}{dx}[x]$$

Here is where the Chain Rule comes into play. Notice that the left-hand derivative is taken *with respect to* x, while the expression we are differentiating *is not a formula of* x. We know that y *depends on* x — so y is really a function of x. Therefore, the Chain Rule tells us

$$\begin{aligned} \frac{d}{dx}\left[y^5\right] = \frac{d}{dx}[x] \quad &\Longrightarrow \quad 5y^4\frac{dy}{dx} = 1 \\ &\Longrightarrow \quad \frac{dy}{dx} = \frac{1}{5y^4} \end{aligned}$$

Since we have assumed that $y = x^{1/5}$, we see that

$$\begin{aligned} \frac{d}{dx}\left[x^{1/5}\right] &= \frac{dy}{dx} \\ &= \frac{1}{5\left(x^{1/5}\right)^4} \\ &= \frac{1}{5x^{4/5}} \end{aligned}$$

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Problem 4. By rewriting $y = x^{1/4}$ as $y^4 = x$, use the technique from the previous example to compute $\frac{d}{dx}\left[x^{1/4}\right]$.

Example 3 *Use the Chain Rule to differentiate the function* $f(x) = x^{5/3}$.

Solution. If we let $y = x^{5/3}$, then we want to compute $\frac{dy}{dx}$. If we assume $y = x^{5/3}$, then it is also true that $y^3 = x^5$. Now, observe

$$\begin{aligned} y^3 = x^5 \quad &\Longrightarrow \quad \frac{d}{dx}\left[y^3\right] = \frac{d}{dx}\left[x^5\right] \\ &\Longrightarrow \quad 3y^2\frac{dy}{dx} = 5x^4 \\ &\Longrightarrow \quad \frac{dy}{dx} = \frac{5x^4}{3y^2} \\ &\Longrightarrow \quad \frac{dy}{dx} = \frac{5x^4}{3(x^{5/3})^2} \\ &\Longrightarrow \quad \frac{dy}{dx} = \frac{5x^{12/3}}{3x^{10/3}} \\ &\Longrightarrow \quad \frac{dy}{dx} = \frac{5x^{2/3}}{3} \end{aligned}$$

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Problem 5. By rewriting $y = x^{-4/5}$ as $y^5 = x^{-4}$, use the technique from the previous example to compute $\frac{d}{dx}\left[x^{-4/5}\right]$.

It is worth noting that there is a pattern appearing in the derivatives we have been computing — one that probably does not surprise you.

$$\frac{d}{dx}\left[x^{1/5}\right] = \frac{1}{5x^{4/5}} = \frac{1}{5}x^{-4/5} = \frac{1}{5}x^{1/5-1}$$

$$\frac{d}{dx}\left[x^{5/3}\right] = \frac{5x^{2/3}}{3} = \frac{5}{3}x^{5/3-1}$$

The Power Rule extends to rational powers of x, and this fact can be proven using the technique outlined in the previous examples and problems.

SUPER POWER RULE

- If r is any rational number, then $\frac{d}{dx}\left[x^r\right] = rx^{r-1}$.

Problem 6. Differentiate the function $f(x) = \sin\left(x^{5/6}\right)$.

HOMEWORK:

- Section 3.1 Page 180 Problems 11, 13, 16, 19
- Section 3.2 Page 188 Problems 20, 21, 28
- Section 3.4 Page 204 Problems 9, 10, 21

Example 4 *Develop a formula for* $\frac{dy}{dx}$ *for the relation* $\cos(y) = \sin(x)$.

Solution. We want to differentiate both sides of this formula with respect to the variable x. Observe

$$\begin{aligned} \cos(y) = \sin(x) \quad &\Longrightarrow \quad \frac{d}{dx}[\cos(y)] = \frac{d}{dx}[\sin(x)] \\ &\Longrightarrow \quad -\sin(y)\frac{dy}{dx} = \cos(x) \\ &\Longrightarrow \quad \frac{dy}{dx} = -\frac{\cos(x)}{\sin(y)} \end{aligned}$$

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Since there is no easy way to rewrite the expression $\sin(y)$ in terms of x, we did not try to simplify the derivative. When we differentiate both sides of a formula with respect to a variable and then solve for resulting derivatives, we refer to this process as *implicit differentiation.*

Problem 7. Use the Chain Rule to develop a formula for $\frac{dy}{dx}$ if $x^3 = \tan(y)$.

Problem 8. Use the Product Rule and the Chain Rule to develop a formula for $\frac{dy}{dx}$ if $y^3 \cos(x) = 10$.

HOMEWORK: Section 3.5 Page 215 Problems 5, 6, 7, 8, 10, 11

Example 5 *Use implicit differentiation to find a formula for* $\dfrac{d}{dx}[\ln(x)]$.

Solution. If we let $y = \ln(x)$, this tells us that $e^y = x$. Observe

$$\begin{aligned}
\frac{d}{dx}[e^y] = \frac{d}{dx}[x] \quad &\Longrightarrow \quad e^y \frac{dy}{dx} = 1 \\
&\Longrightarrow \quad \frac{dy}{dx} = \frac{1}{e^y} \\
&\Longrightarrow \quad \frac{d}{dx}[\ln(x)] = \frac{1}{e^{\ln(x)}} \\
&\Longrightarrow \quad \frac{d}{dx}[\ln(x)] = \frac{1}{x}
\end{aligned}$$

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Example 6 *Differentiate the function* $g(x) = \ln\left(x^{3/7} - 4\cos(x)\right)$.

Solution. Let $u(x) = x^{3/7} - 4\cos(x)$ and let $f(u) = \ln(u)$. Observe

$$\begin{aligned}
\frac{dg}{dx} &= \frac{df}{du} \cdot \frac{du}{dx} \\
&= \frac{d}{du}[\ln(u)] \cdot \frac{d}{dx}\left[x^{3/7} - 4\cos(x)\right] \\
&= \frac{d}{du}[\ln(u)] \cdot \left(\frac{d}{dx}\left[x^{3/7}\right] - 4\frac{d}{dx}[\cos(x)]\right) \\
&= \frac{1}{u} \cdot \left(\frac{3}{7}x^{-4/7} + 4\sin(x)\right) \\
&= \frac{1}{x^{3/7} - 4\cos(x)}\left(\frac{3}{7}x^{-4/7} + 4\sin(x)\right)
\end{aligned}$$

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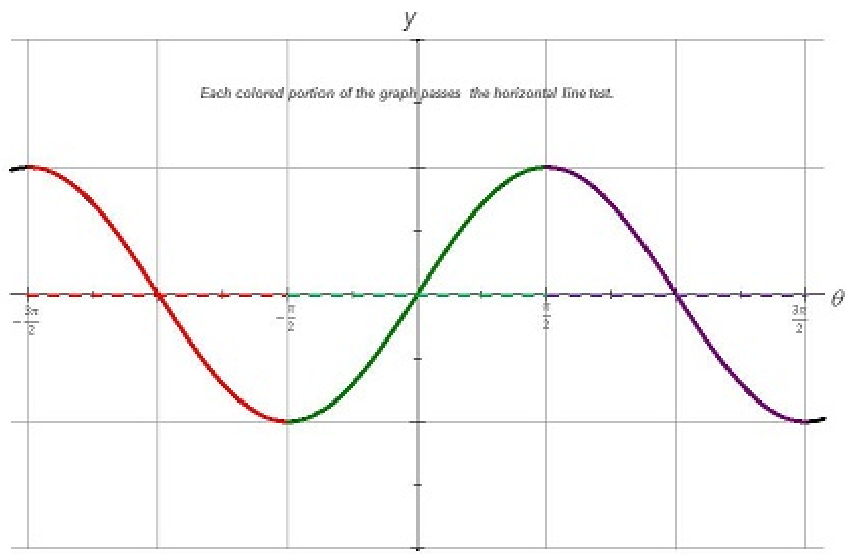
Problem 9. Differentiate the function $g(x) = \sin(\ln(x))$.

Problem 10. Differentiate the function $g(x) = \ln\left(3x + \sqrt{x}\right)$.

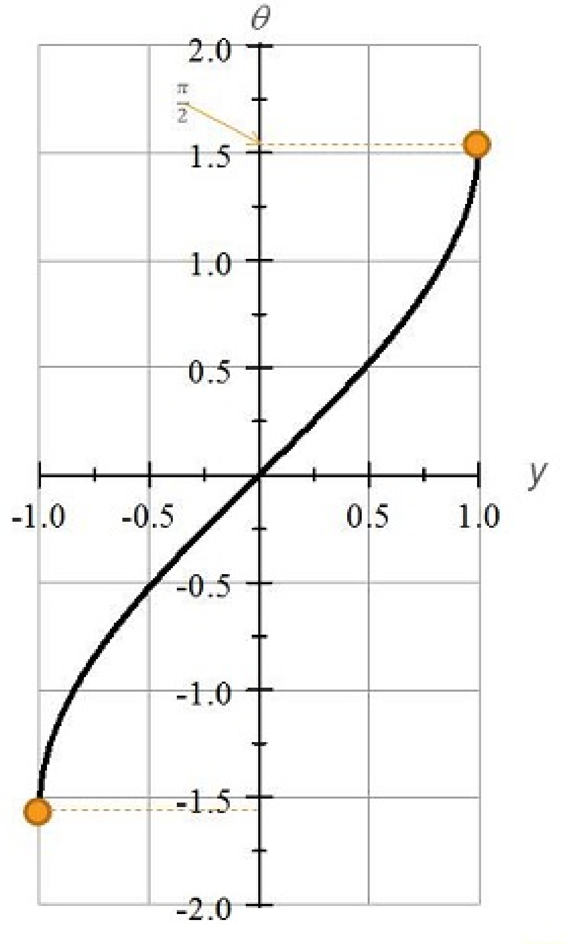
HOMEWORK: Section 3.6 Page 223 Problems 2, 3, 5, 10, 11

We used the fact that $f(x) = e^x$ serves as the inverse of the function $g(x) = \ln(x)$ to help develop the derivative function for g. There is no function that reverses the sine function over all of its domain; consequently, the sine function does not have a true inverse like the natural logarithm function does. However, in physics and engineering, it is still important to be able to solve equations involving the sine function. We accomplish this using a *partial inverse* for the sine function.

The reason the sine function $y = f(\theta) = \sin(\theta)$ does not have a true inverse is because its graph fails the horizontal line test. However, as the diagram below shows, there are many input intervals where the graph of the sine function *does* pass the horizontal line test.



We could define a *partial inverse function* for the sine function on any one of these intervals. It is customary to use the interval $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$ to define a partial inverse. The partial inverse defined on this particular interval is called the *principal inverse sine* function; or more commonly, the *arcsine* function. We obtain the graph of the arcsine function by switching the roles of input and output variable (switching the horizontal and vertical axes) for the sine graph.



The function $\theta = g(y) = \arcsin(y)$ is defined by the graph shown above. This function reverses the sine function, but only on the interval $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$. In other words,

$$\arcsin(\sin(\theta)) = \theta$$

only when $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$.

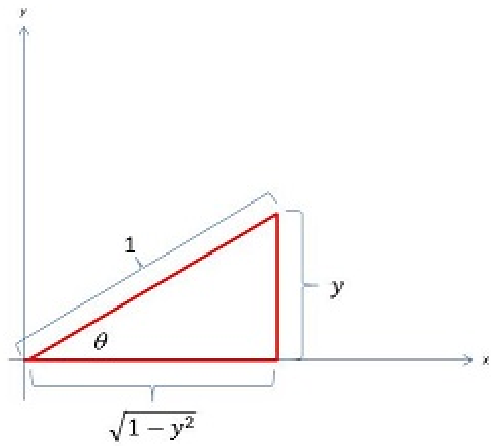
FUNDAMENTAL RELATIONSHIP BETWEEN SINE AND ARCSINE

Whenever $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$, we have $y = \sin(\theta)$ if and only if $\arcsin(y) = \theta$

This relationship is sufficient for us to develop a formula for the derivative function of $\theta = \arcsin(y)$. To this end, suppose that $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$. We know that $\theta = \arcsin(y)$ implies that $y = \sin(\theta)$. Observe

$$\begin{aligned}
\frac{d}{dy}[y] = \frac{d}{dy}[\sin(\theta)] \quad &\Longrightarrow \quad 1 = \cos(\theta) \cdot \frac{d\theta}{dy} \\
&\Longrightarrow \quad \frac{1}{\cos(\theta)} = \frac{d\theta}{dy} \\
&\Longrightarrow \quad \sec(\theta) = \frac{d\theta}{dy} \\
&\Longrightarrow \quad \sec(\arcsin(y)) = \frac{d}{dy}[\arcsin(y)]
\end{aligned}$$

Now, it is important to remember that $\theta = \arcsin(y)$ is an angle measure between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$. In particular, this is the measure of an angle in a right triangle having one vertex at the origin and one leg on the positive x-axis. Furthermore, since $y = \sin(\theta)$, we have the following diagram.

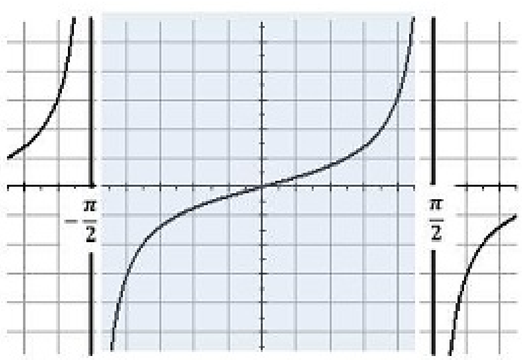


Consequently, we know that

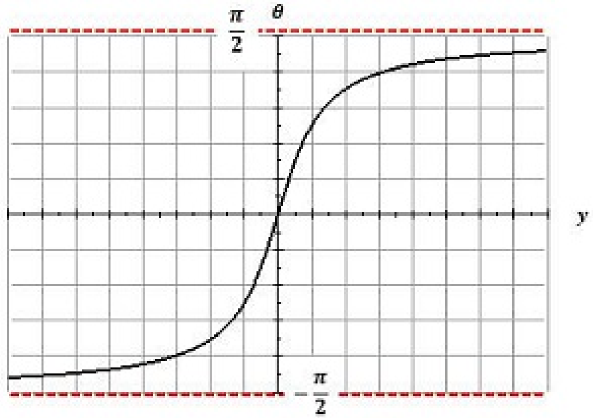
$$\begin{aligned} \frac{d}{dy}\left[\arcsin(y)\right] &= \sec(\arcsin(y)) \\ &= \sec(\theta) \\ &= \frac{1}{\cos(\theta)} \\ &= \frac{1}{\sqrt{1-y^2}} \end{aligned}$$

It is possible to define partial inverse functions for each of the other five trigonometric functions. However, only the partial inverse function for the tangent function is often encountered outside of mathematics. The principal inverse tangent (or *arctangent*) function has a similar definition to that of the arcsine function.

As with the arcsine function, the arctangent function is defined by selecting an input interval where the graph of the tangent function passes the horizontal line test. The diagram below shows the input interval used.



The arctangent function is defined by forming the inverse function for the tangent function on this interval. As with the arcsine function, the arctangent function is defined from its graph. The graph of the arctangent function is obtained by switching the input and output variables for the tangent function on this interval.



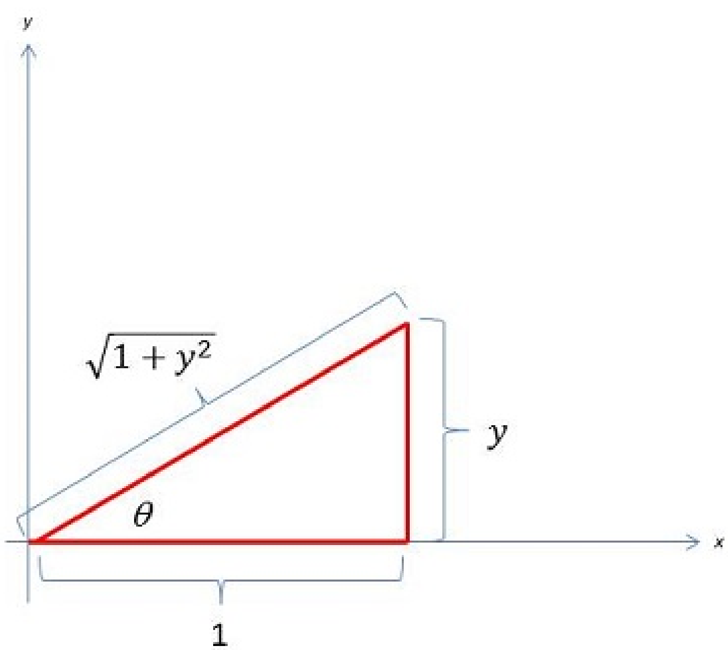
FUNDAMENTAL RELATIONSHIP BETWEEN TANGENT AND ARCTANGENT

Whenever $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$, we have $y = \tan(\theta)$ if and only if $\arctan(y) = \theta$

We can develop a formula for the derivative of the arctangent function by following the same procedure we followed for the arcsine function. Let $\theta = \arctan(y)$ and suppose that $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$. We know that $\theta = \arctan(y)$ implies that $y = \tan(\theta)$. Let $u(\theta) = \theta$, and let $f(u) = \tan(u)$. Observe

$$\begin{aligned}
\frac{d}{dy}[y] = \frac{d}{dy}[\tan(\theta)] \quad &\Longrightarrow \quad \Longrightarrow 1 = \sec^2(\theta) \cdot \frac{d\theta}{dy} \\
&\Longrightarrow \quad \frac{1}{\sec^2(\theta)} = \frac{d\theta}{dy} \\
&\Longrightarrow \quad \cos^2(\theta) = \frac{d\theta}{dy} \\
&\Longrightarrow \quad \cos^2(\arctan(y)) = \frac{d}{dy}[\arctan(y)]
\end{aligned}$$

Now, it is important to remember that $\theta = \arctan(y)$ is an angle measure between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$. In particular, this is the measure of an angle in a right triangle having one vertex at the origin and one leg on the positive x-axis. Furthermore, since $y = \tan(\theta)$, we have the following diagram.



Consequently, we know that

$$
\begin{aligned}
\frac{d}{dy}\left[\arctan(y)\right] &= \cos^2(\arctan(y)) \\
&= \cos^2(\theta) \\
&= \left(\frac{1}{\sqrt{1+y^2}}\right)^2 \\
&= \frac{1}{1+y^2}
\end{aligned}
$$

Problem 11. Use the Chain Rule to differentiate $g(x) = \arcsin(4x)$.

Problem 12. Use the Product Rule and the Chain Rule to differentiate $g(x) = x^2 \arctan(x^3)$.

HOMEWORK: Section 3.5 Page 216 Problems 49, 50, 51, 56, 57 (Your book uses $\sin^{-1}(y)$ to denote the arcsine function and $\tan^{-1}(y)$ to denote the arctangent function.)