We say that a function $y = f(x)$ is locally linear at a point $(a, f(a))$ provided "zooming in" on the graph of f at the point $(a, f(a))$ produces a view of the graph that looks more and more like a straight line. When a function is locally linear, we know several things:

1. Near $x = a$, small changes in the value of the input x will be approximately proportional to corresponding changes in the output value of f. In other words, for values of x close to $x = a$, we have
$$\Delta f(x) \approx m\Delta x$$
for some constant m. This constant m is called the local constant rate of change for $f(x)$ with respect to x near $x = a$.

2. It is possible to construct a tangent line to the graph of f at the point $(a, f(a))$. The point-slope formula for this line is
$$y = m(x - a) + f(a)$$
where m is the local constant rate of change for $f(x)$ with respect to x near $x = a$. For values of x close to $x = a$, the output from the tangent line is a good approximation to the output of f.

3. The instantaneous rate of change of the output values $f(x)$ with respect to x for the function f exists at the input value $x = a$. This number, denoted by $f'(a)$, is the limiting value of the average rates of change for the function f over increasingly small input intervals that begin or end at $x = a$. In symbols, we have
$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = f'(a)$$

4. The instantaneous rate of change of the output values $f(x)$ with respect to x for the function f at $x = a$ is the same as the local constant rate of change for $f(x)$ with respect to x near $x = a$.

We have seen that a function f will be locally linear at a point $(a, f(a))$ provided the graph has no jump or very sharp "kink" at this point. If we have a formula for the function f, we can approximate the derivative value $f'(a)$ simply by computing the average rate of change for f on a very small input interval that begins or ends at $x = a$ (provided we know the function is locally linear at $(a, f(a))$).

Example 1. Estimate the value of $f'(2)$ to at least three decimal place accuracy if the function f is defined by
$$f(t) = \frac{\sqrt{t+1}}{\cos(t)}$$

Solution. The best way to attack this problem would be to use a graphing calculator to construct a table of values for the function
$$g_2(h) = \frac{f(2+h) - f(2)}{h}$$

In the table, we want to choose values of h close enough to 0 that the output from g_2 is stable to at least three decimal places. On the graphing calculator, let
$$Y_1(X) = \sqrt{X+1}/\cos(X)$$
$$Y_2(X) = \big(Y_1(2+X) - Y_1(2)\big)/X$$

After some trial and error, you can see that initializing the table at $X = -0.00001$ with an increment of $\Delta X = 0.000001$ gives us outputs from Y_2 that are stable to three decimal places. In particular, we find that the output value

$$g_2(0.000001) \approx 8.4007$$

is an approximation to $f'(2)$ that is accurate to three decimal places.

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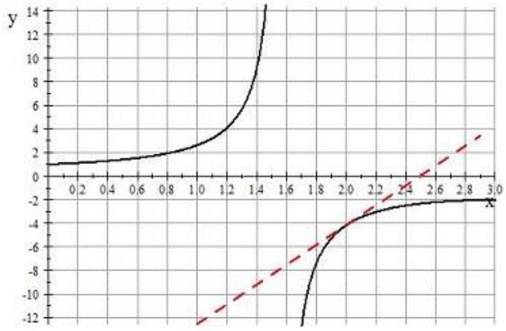
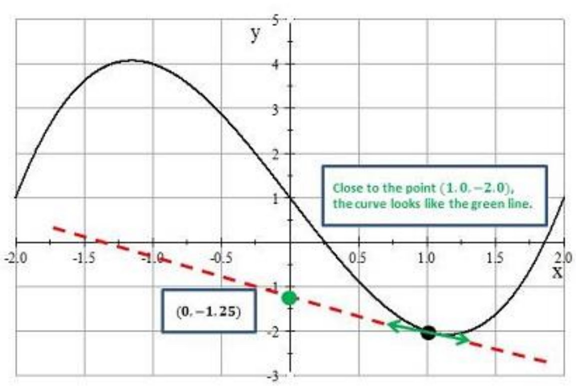
Here is a graph of the function f along with the line $y = 8.4007(x-2) + f(2)$.



On the other hand, if we are given only the graph of a function f, we can estimate the value of $f'(a)$ by sketching the tangent line to the graph of f at the point $(a, f(a))$ and using two points on our sketch to determine the slope of the line. Alternatively, we could select a second point $(b, f(b))$ on the graph of the function f and use the average rate of change for the function f on the input interval from $x = a$ to $x = b$ as an approximation to the value of $f'(a)$. Of course, this approach will only be reliable if we can select $x = b$ very close to $x = a$ and obtain a reasonably accurate value for $f(b)$.

Both methods mentioned in the last paragraph are, of course, prone to substantial error. When attempting to sketch the tangent line, it is important to remember that the slope of this line represents the *local constant rate of change* in the output values $f(x)$ with respect to the input values x near $x = a$. Consequently, the behavior of the curve very close to the point $(a, f(a))$ should be your guide in sketching the tangent line.

Consider, for example, the graph of a function $y = f(x)$ shown below. Suppose we want to estimate the value of $f'(1.0)$.



Based on the sketch of the tangent line drawn in the diagram above, we can see that

$$f'(1.0) \approx \frac{-2.00 - (-1.25)}{1 - 0} = -0.75$$

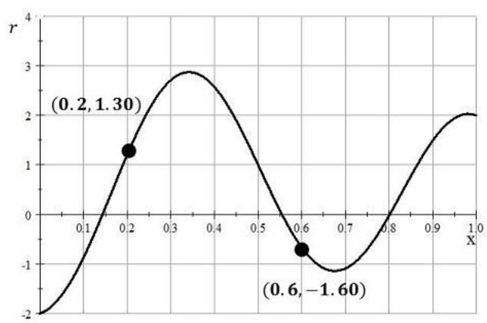
Problem 1. On the graph provided above, sketch the line tangent to the graph of the function f at the point $(-0.5, 2.90)$ and use your sketch to determine the approximate value of $f'(-0.5)$.

Let $y = f(x)$ be a function. The function $r = f'(x)$ is defined to be the function whose output for any input value $x = a$ is the instantaneous rate of change for f with respect to x at $x = a$. This function is called the *derivative* function for the function f. It is also common to use the symbol

$$r = \frac{df}{dx}$$

to represent the derivative function.

Example 2. The diagram below shows the graph of the derivative function $r = f'(x)$ for a function $y = f(x)$. What is the meaning and the value of the numbers $f'(0.2)$ and $f'(0.6)$? What is the point-slope formula for the tangent line to the graph of the function f at the point $(0.2, f(0.2))$?



Solution. The graph tells us that $f'(0.2) = 1.30$. Consequently, we have the following, equivalent interpretations for this number:

1. The instantaneous rate of change for the function f with respect to x at $x = 0.2$ is $r = 1.30$.
2. The local constant rate of change for $f(x)$ with respect to x near $x = 0.2$ is $r = 1.30$.
3. The slope of the line tangent to the graph of f at the point $(0.2, f(0.2))$ is $r = 1.30$.

The point-slope formula for the tangent line to the graph of f at the point $(0.2, f(0.2))$ is

$$y = 1.30(x - 0.2) + f(0.2)$$

Since we have no information about the rule f, we are not able to evaluate $f(0.2)$.

We have the following, equivalent interpretations for the number $f'(0.6)$:

1. The instantaneous rate of change for the function f with respect to x at $x = 0.6$ is $r = -1.60$.
2. The local constant rate of change for $f(x)$ with respect to x near $x = 0.6$ is $r = -1.60$.
3. The slope of the line tangent to the graph of f at the point $(0.6, f(0.6))$ is $r = -1.60$.

The point-slope formula for the tangent line to the graph of f at the point $(0.6, f(0.6))$ is

$$y = -1.60(x - 0.6) + f(0.6)$$

Since we have no information about the rule f, we are not able to evaluate $f(0.6)$.

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Problem 2. Use the graph of the derivative function for the function f provided in Example 2 to help you construct the point-slope formula for the line tangent to the graph of the function f at the point $(0.9, f(0.9))$.

Example 3. Consider the function $y = f(x) = x^3 - 4x + 1$. Sketch the graph of the derivative function $r = f'(x)$.

Solution. One way to accomplish this task would be to estimate output values for the derivative function f' for some input values of x, plot the ordered pairs $(x, f'(x))$ and see if we can detect a pattern. Since

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

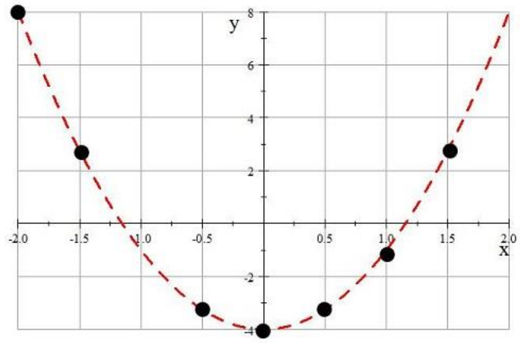
we can compute very good estimates for the values of $f'(a)$ by computing the average rate of change for f on the interval $a \le x \le a + 0.001$. For example,

$$f'(-2) \approx \frac{f(-1.999) - f(-2)}{0.001} \approx 7.994$$

Using this approximation method, we can construct the following table.

Input Value x	-2.00	-1.50	-0.50	0	0.50	1.00	1.50
Approximation for $f'(x)$	7.994	2.746	-3.251	-4.000	-3.248	-0.997	2.755

Plotting these ordered pairs and connecting them with a smooth curve gives us an approximation to the actual graph of the derivative function.



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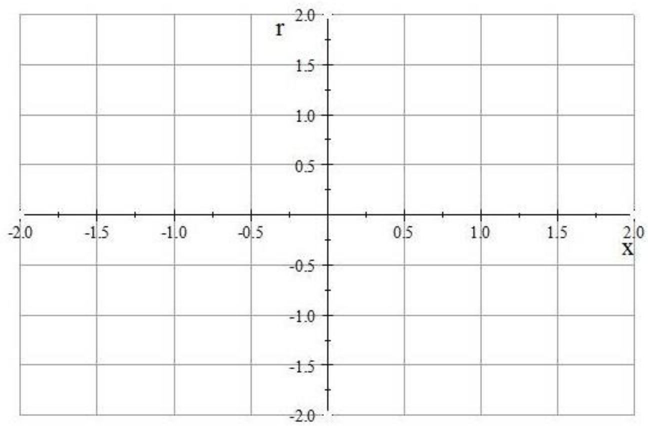
Problem 3. Use the method in Example 3 to construct a sketch of the derivative function $r = f'(x)$ for the function

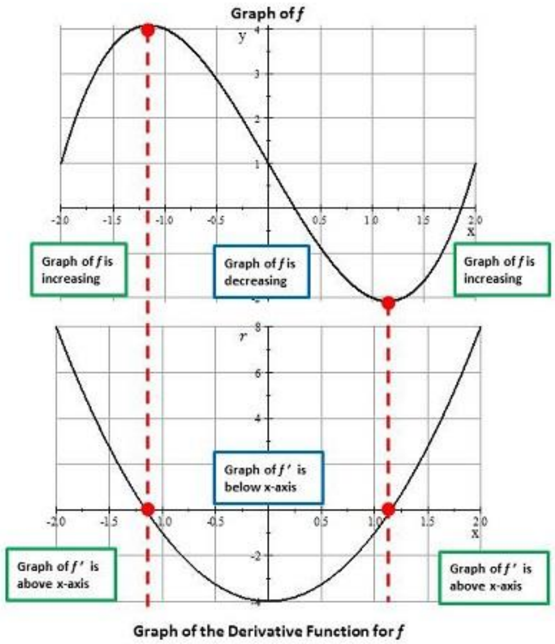
$$y = f(x) = \frac{1 + \cos(x)}{1 + x^2}$$

Input Value x	-2.00	-1.50	-0.50	-0.25	0.00	0.25	1.00	1.50	2.00
Approximation for $f'(x)$									



Problem 4. Use your sketch from Problem 3 to estimate the value of $f'(0.75)$ and then construct the approximate formula for the line tangent to the graph of the function f in Problem 3 at the point $(0.75, f(0.75))$.

There is an important relationship between the graph of a function and the graph of its derivative function. Consider the diagram below which shows the graph of the function $y = f(x) = x^3 - 4x + 1$ along with the graph of its derivative function.



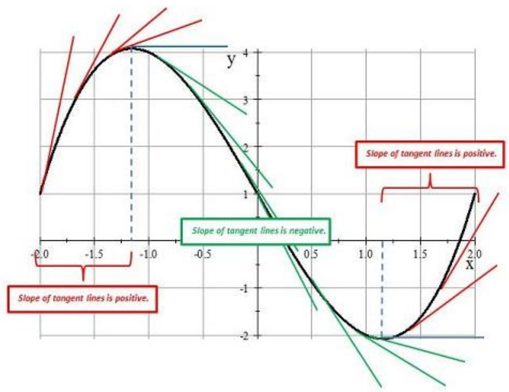
Suppose that $y = f(x)$ is a function that is locally linear at each input value in its domain. The diagram above highlights the relationship between the graph of the function f and the graph of its derivative function.

- Suppose the graph of f is decreasing on an input interval, and let $x = a$ be any input value from this interval. We can choose a value of h close enough to 0 that the average rate of change for f on the interval from $x = a$ to $x = a + h$ will be *negative*. Therefore, the value of $f'(a)$ will be negative. Consequently, the graph of the function f' will be below the input axis on this interval.

- Suppose the graph of f is increasing on an input interval, and let $x = a$ be any input value from this interval. We can choose a value of h close enough to 0 that the average rate of change for f on the interval from $x = a$ to $x = a + h$ will be *positive*. Therefore, the value of $f'(a)$ will be positive. Consequently, the graph of the function f' will be above the input axis on this interval.

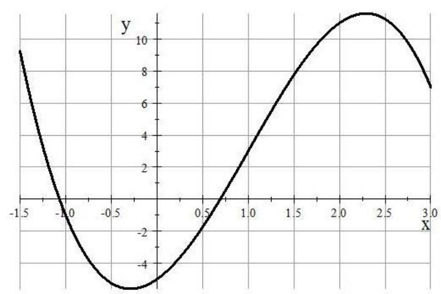
- If the graph of the function f has a turning point at the input value $x = a$, then we will have $f'(a) = 0$.

We can also understand the relationship between the graph of a function f and its derivative function by thinking about tangent lines to the graph of the function f.

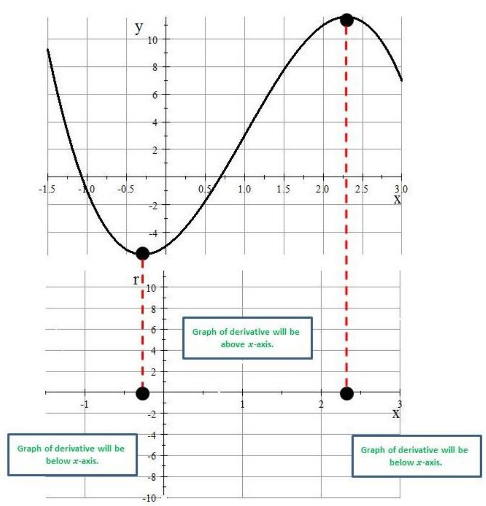


This approach is particularly helpful if we are trying to sketch the graph of the derivative function for a function $y = f(x)$ when we have only the graph of the function f to go on.

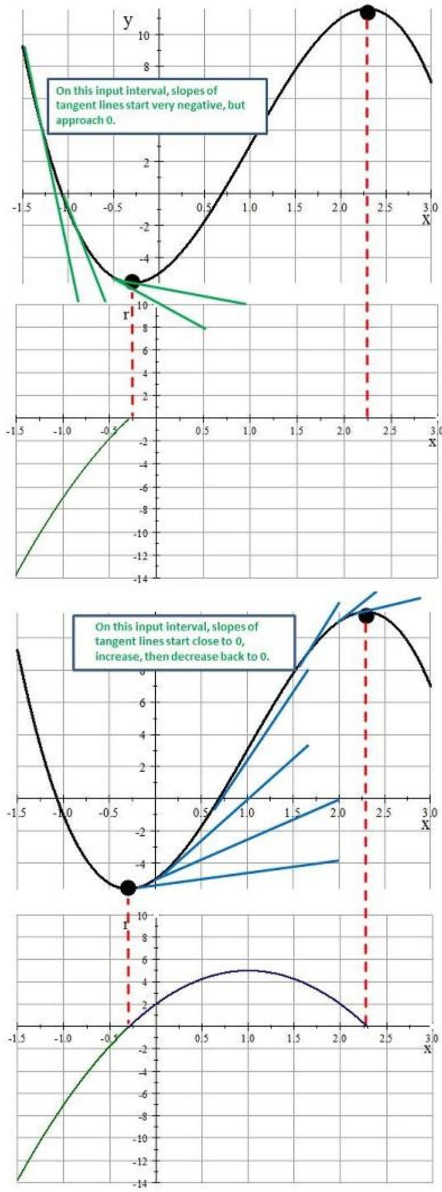
Example 4. The graph of a function $y = f(x)$ is given below. Use this graph to construct a sketch of the derivative function $r = f'(x)$.



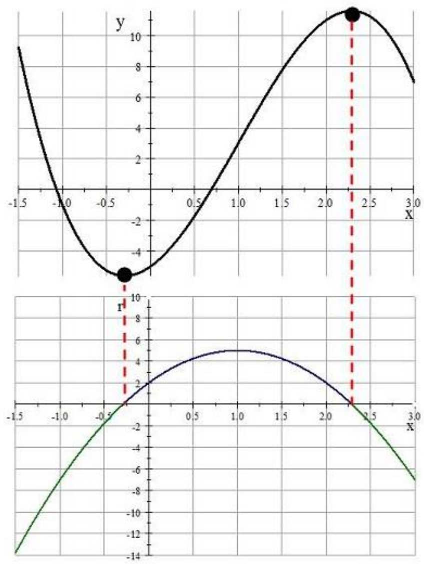
Solution. The easiest way to start would be to locate the turning points for the graph of the function f, because the input values for these points determine where the input intervals of increase and decrease for the function f occur, and they determine where the graph of the derivative function will cross the input axis.



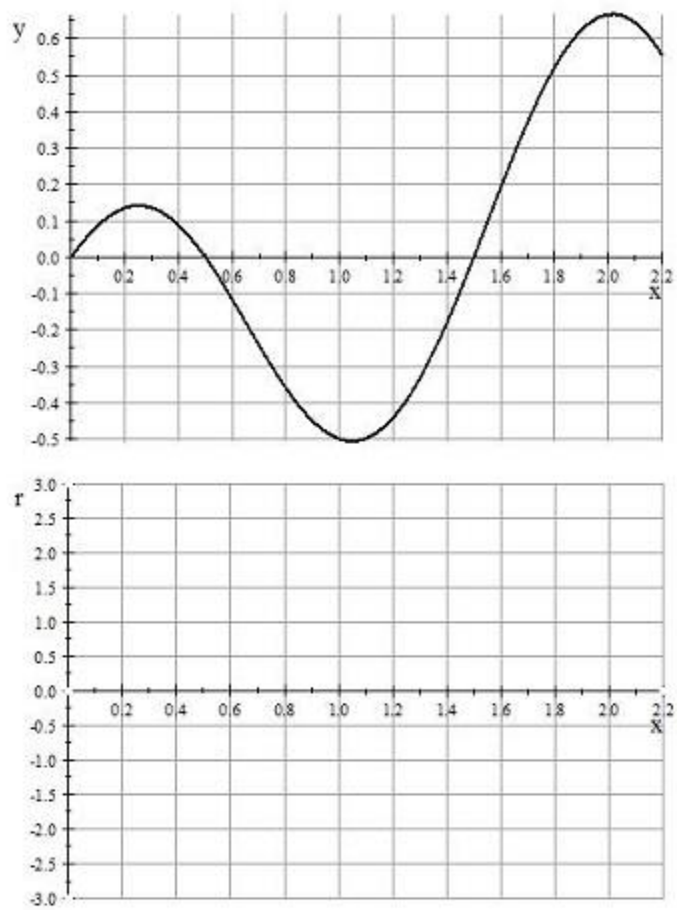
Now, we can sketch some tangent lines to the graph of the function f at various points in each interval to get an idea about how the graph of the derivative function is behaving. We could even estimate the slopes of these tangent lines to determine approximate output values of the derivative function if we wished.



On the last input interval, the slopes of tangent lines will start close to 0 (but negative), and become more negative. Consequently, the completed sketch of the derivative function will look like the following diagram.



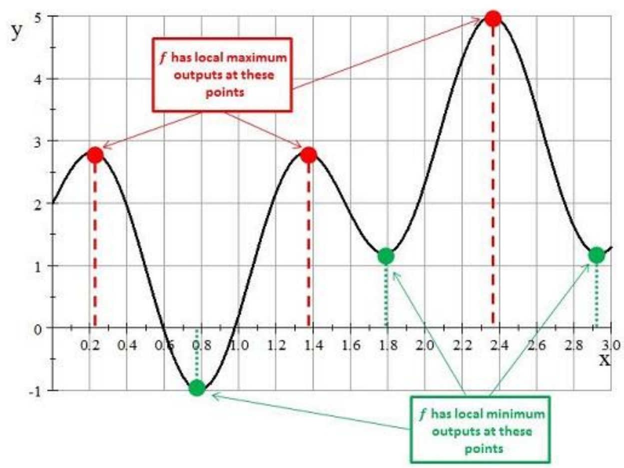
Problem 5. The graph of a function $y = f(x)$ is shown below. Use this graph to sketch an approximation for the graph of the derivative function for f.



Let $y = f(x)$ be a function.

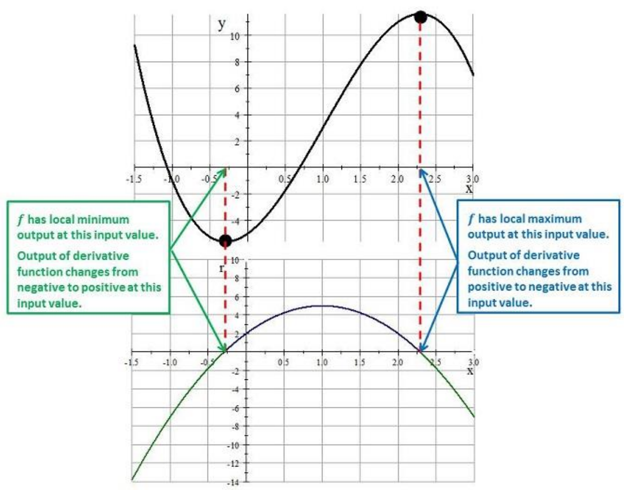
- We say that the function f has a *local maximum output* at an input value $x = a$ provided $f(a)$ is the largest output for f on some small input interval containing $x = a$.
- We say that the function f has a *local minimum output* at an input value $x = a$ provided $f(a)$ is the smallest output for f on some small input interval containing $x = a$.

We often say that the function f has a *local extremum* at an input value $x = a$ when we know that f has a local maximum or minimum output at $x = a$ but have not yet determined which one it is.

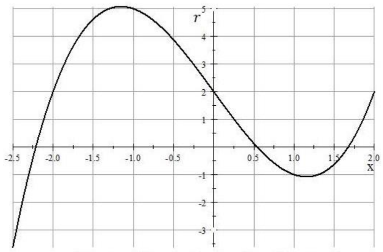


The diagram above shows the graph of a function $y = f(x)$ and identifies the input values where the function f has local extrema. For example, the function f has a local maximum output at the input values $x \approx 0.79$, $x \approx 1.39$, and $x \approx 2.39$.

There is a close connection between the derivative function for a function $y = f(x)$ and the existence of local extrema for the function f. We can see this connection by returning to the function we explored in Example 4.



Example 5. The graph of the *derivative function* for a function $y = f(x)$ is shown below. Based on this graph, at what input values does the function f have local maximum or local minimum output?



Graph of the derivative function for f

Solution. To answer this question, we use the graph to determine the input values where the output of the derivative function changes from positive to negative or vice-versa. Looking at the graph, we see that the output of the derivative function changes from negative to positive at the input value $x \approx -2.20$ and $x \approx 1.60$. Consequently, the function f will have local *minimum* output at these input values. On the other hand, the output of the derivative function changes from positive to negative at the input value $x \approx 0.55$; hence, we know that the function f will have a local *maximum* output at this input value.

Since we have no information about the rule defining the function f, we cannot determine what the values of the local maximum and minimum outputs will be.

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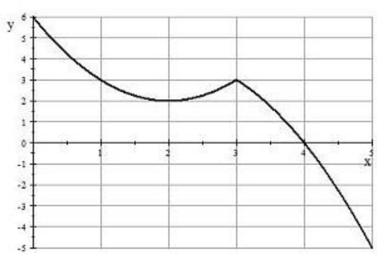
First Derivative Test

Let $y = f(x)$ be a function and let $r = f'(x)$ represent its derivative function. Suppose that $f(a)$ exists.

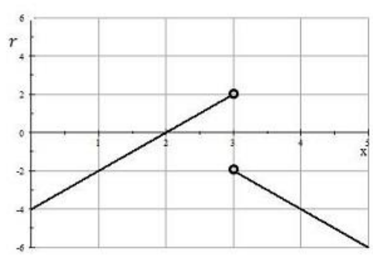
- If the output of f' changes from positive to negative at $x = a$, then the function f has a local maximum output at $x = a$.
- If the output of f' changes from negative to positive at $x = a$, then the function f has a local minimum output at $x = a$.

(This test assumes we are reading the graph from left to right.)

The First Derivative Test can be used even if the derivative function is undefined at $x = a$. For example, the function $y = f(x)$ shown below has a local maximum output at the input value $x = 3$, but the function f is *not* locally linear at the point $(3, f(3))$. Nonetheless, the output of the derivative function $r = f'(x)$ changes from positive to negative at $x = 3$.



Graph of f



Graph of f'

Homework: Section 2.8 (Pages 161 – 162) Problems 3, 4, 5, 6, 8, 9, 10, 12, 13, 14, 15
Section 4.3 (Pages 300 – 301) Problems 5, 6