Suppose that $y = f(x)$ is a function. If the function f is locally linear at the point $(a, f(a))$, then it is common to say that the function f is *differentiable* at the input value $x = a$. The various processes we use to determine the derivative function for f are known collectively as *differentiation*.

Differentiating a function $y = f(x)$ amounts to determining the derivative function $r = f'(x)$. To do this, we must either

(1) Construct the graph of the derivative function f' from the formula or graph for the function f
(2) Find a way to determine a formula for the derivative function f' directly from its limit definition

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

Determining a formula from the limit definition requires us to show that the average rate of change function

$$g_x(h) = \frac{f(x+h) - f(x)}{h}$$

has a removable discontinuity at the input value $h = 0$; and if g_x is an algebraic function, this can usually be done using basic algebra.

Example 1. Differentiate the function $y = f(t) = t^{-1}$ with respect to the input variable t.

Solution. For any fixed value of the input variable t, consider the average rate of change function

$$g_t(h) = \frac{f(t+h) - f(t)}{h} = \left(\frac{1}{h}\right)\left(\frac{1}{t+h} - \frac{1}{t}\right)$$

We want to use algebra to simplify the rightmost formula. The goal of the simplification is to show that the factor $1/h$ can be cancelled from the formula. Observe

$$\begin{aligned}
g_x(h) &= \left(\frac{1}{h}\right)\left(\frac{1}{t+h} - \frac{1}{t}\right) \\
&= \left(\frac{1}{h}\right)\left(\frac{1}{t+h}\left[\frac{t}{t}\right] - \frac{1}{t}\left[\frac{t+h}{t+h}\right]\right) \\
&= \left(\frac{1}{h}\right)\left(\frac{t - (t+h)}{t(t+h)}\right) \\
&= \left(\frac{1}{h}\right)\left(-\frac{h}{t(t+h)}\right) \\
&= -\frac{1}{t(t+h)} \qquad\qquad (h \neq 0)
\end{aligned}$$

Now, having shown that the function g_t has a removable discontinuity at the input value $h = 0$, we can determine the formula for the derivative function for f. Observe

$$f'(t) = \lim_{h\to 0} \frac{f(t+h) - f(t)}{h} = \lim_{h\to 0} \frac{-1}{t(t+h)} = -\frac{1}{t^2}$$

****

The derivative function for a function $y = f(x)$ with respect to the input variable x is often denoted by the operation symbol

$$\frac{d}{dx}[\, f(x)\,]$$

This symbol means "the derivative function with respect to *x* of the function appearing in brackets." For example, we can now write

$$\frac{d}{dt}[t^{-1}] = -\frac{1}{t^2}$$

It is worth pointing out that the derivative formula is *invariant* regarding how we choose to name the input variable. For example, it will also be true that

$$\frac{d}{dx}[x^{-1}] = -\frac{1}{x^2} \qquad \text{and} \qquad \frac{d}{da}[a^{-1}] = -\frac{1}{a^2}$$

Example 2. Determine $\frac{d}{dz}[z^2]$.

Solution. We are asked to determine the derivative function with respect to the input variable z for the function $f(z) = z^2$. Once again, consider the average rate of change function

$$g_z(h) = \frac{f(z+h) - f(z)}{h} = \frac{(z+h)^2 - z^2}{h}$$

We can use basic algebra to show that this function has a removable discontinuity at the input value $h = 0$. Observe

$$\begin{aligned} g_z(h) &= \frac{z^2 + 2zh + h^2 - z^2}{h} \\ &= \frac{2zh + h^2}{h} \\ &= 2z + h \qquad\qquad (h \neq 0) \end{aligned}$$

Now that we have demonstrated that the function g_z has a removable discontinuity at the input value $h = 0$, we can determine the formula for the derivative function for $f(z) = z^2$. Observe

$$f'(z) = \frac{d}{dz}[z^2] = \lim_{h\to 0}(2z + h) = 2z$$

****

Example 3. Differentiate the function $b = f(w) = w^{1/2}$ with respect to the input variable w.

Solution. Once again, consider the average rate of change function

$$g_w(h) = \frac{f(w+h) - f(w)}{h} = \frac{(w+h)^{1/2} - w^{1/2}}{h}$$

Proving that this function has a removable discontinuity at the input value $h = 0$ requires a bit more ingenuity than we needed in the last two examples, because we cannot "expand" a binomial raised to a fractional power the same way we can expand a binomial raised to an integer power.

The key to simplifying in this case is to make the observation that, thanks to the laws of exponents, we know for any expressions A and B, the following equation is true as long as $A^{1/2}$ and $B^{1/2}$ are defined:

$$\left(A^{1/2} - B^{1/2}\right)\left(A^{1/2} + B^{1/2}\right) = A - B$$

With this in mind, observe that

$$\begin{aligned}
g_w(h) &= \frac{(w+h)^{1/2} - w^{1/2}}{h} \\
&= \left[\frac{(w+h)^{1/2} - w^{1/2}}{h}\right]\left[\frac{(w+h)^{1/2} + w^{1/2}}{(w+h)^{1/2} + w^{1/2}}\right] \\
&= \left(\frac{1}{h}\right)\left[\frac{(w+h) - w}{(w+h)^{1/2} + w^{1/2}}\right] \\
&= \left(\frac{1}{h}\right)\left[\frac{h}{(w+h)^{1/2} + w^{1/2}}\right] \\
&= \frac{1}{(w+h)^{1/2} + w^{1/2}} \qquad (h \neq 0)
\end{aligned}$$

Now that we have demonstrated that the function g_w has a removable discontinuity at the input value $h = 0$, we can determine for the formula for the derivative function for $f(w) = w^{1/2}$. Observe

$$f'(w) = \frac{d}{dw}\left[w^{1/2}\right] = \lim_{h\to 0}\frac{1}{(w+h)^{1/2} + w^{1/2}} = \frac{1}{w^{1/2} + w^{1/2}} = \frac{1}{2w^{1/2}}$$

****

Problem 1. Follow the method used in Examples 1 – 3 to determine $\frac{d}{dp}[p^{-2}]$.

Problem 2. Differentiate the function $y = f(s) = s^3$ following the method used in Examples 1 – 3.
Hint: Remember that $(s+h)^3 = s^3 + 3s^2h + 3sh^2 + h^3$.

The algebra appearing in the previous examples and problems is daunting. However, there is a surprising pattern that appears in the formulas for the derivative functions we have constructed. Observe

$$f(x) = x^{-1} \quad \Rightarrow \quad f'(x) = -\frac{1}{x^2} = (-1)x^{-2} = (-1)x^{-1-1}$$

$$f(x) = x^{2} \quad \Rightarrow \quad f'(x) = 2x = (2)x^{1} = (2)x^{2-1}$$

$$f(x) = x^{1/2} \quad \Rightarrow \quad f'(x) = \frac{1}{2x^{1/2}} = \left(\frac{1}{2}\right)x^{-1/2} = \left(\frac{1}{2}\right)x^{1/2-1}$$

$$f(x) = x^{-2} \quad \Rightarrow \quad f'(x) = -\frac{2}{x^3} = (-2)x^{-3} = (-2)x^{-2-1}$$

$$f(x) = x^{3} \quad \Rightarrow \quad f'(x) = 3x^2 = (3)x^{3-1}$$

Power Rule for Differentiation

If q is any rational number, then the derivative function for the function $y = f(x) = x^q$ is the function defined by

$$r = f'(x) = qx^{q-1}$$

Problem 3. Use the Power Rule to help you construct the point-slope formula for the line tangent to the graph of the function $y = x^{-4}$ at the point $(2, f(2))$.

Problem 4. Consider the function $y = f(x) = x^{3/2}$. Are there any values of x where $f'(x) = 6$?

The Power Rule for Differentiation is an example of a *specific derivative formula*. The Power Rule summarizes a pattern seen whenever we differentiate a power function. Here is another specific derivative formula:

Constant Function Rule for Differentiation

If $y = f(x)$ is any constant function, then the derivative function for f is the function defined by $r = f'(x) = 0$.

It is not hard to see why this specific derivative formula should be valid, since any constant function is a linear function whose slope is 0. Because of this, for any change in the values of the input variable, the change in the output variable will be 0. Consequently, the average rate of change for a constant function on any input interval will always be 0.

Now, let's consider the *exponential* functions. An exponential function has the form $y = f(x) = B^x$, where B is a positive constant. Exponential functions are transcendental functions; consequently, working with the average rate of change function for an exponential function will pose special challenges. Consider the average rate of change function

$$g_x(h) = \frac{f(x+h) - f(x)}{h} = \frac{B^{x+h} - B^x}{h}$$

Once again, the goal is to prove that this function has a removable discontinuity at $h = 0$. However, this time basic algebra will be of limited use. The input variable h in the numerator is trapped in the exponent, and no amount of algebra will be able to undo this fact. Let's see what basic algebra is able to tell us:

$$g_x(h) = \frac{B^{x+h} - B^x}{h} = \frac{B^x B^h - B^x(1)}{h} = B^x \cdot \frac{B^h - 1}{h}$$

In this situation, we are not able to prove using algebra that the function $g_x(h)$ has a removable discontinuity at the input value $h = 0$. However, we have been able to establish something of interest. Observe that

$$\frac{d}{dx}[B^x] = \lim_{h\to 0} B^x \cdot \frac{B^h - 1}{h} = B^x \cdot \lim_{h\to 0} \frac{B^h - 1}{h}$$

Basic algebra (the laws of exponents in particular) can tell us that the derivative function for $f(x) = B^x$ will be a *constant times the function* f. In particular,

$$f'(x) = f(x) \cdot \lim_{h\to 0} \frac{B^h - 1}{h}$$

There are ways to determine the exact value of this limit process; however, these methods are beyond the scope of this course. Therefore, instead of tackling the limit process directly, let's look at estimates of this limit process for various values of B and see if there is a pattern.

For example, let's consider the base $B = 2$. We know that

$$\lim_{h\to 0} \frac{2^h - 1}{h} \approx \frac{2^{0.0001} - 1}{0.0001} \approx 0.69317$$

Consequently, we know that

$$\frac{d}{dx}[2^x] \approx 0.69317 \cdot 2^x$$

We can use this technique to approximate the constant for other values of B. For example,

$$\lim_{h\to 0} \frac{0.5^h - 1}{h} \approx -0.69317 \qquad\qquad \lim_{h\to 0} \frac{4^h - 1}{h} \approx 1.38634$$

$$\lim_{h\to 0}\frac{3^h-1}{h}\approx 1.0986 \qquad \lim_{h\to 0}\frac{0.25^h-1}{h}\approx -1.38634$$

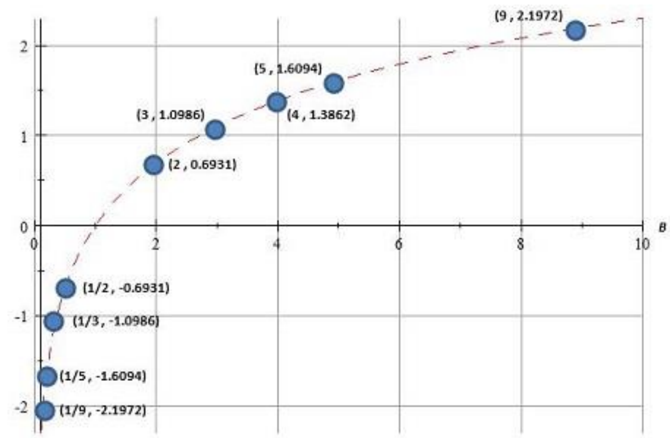
$$\lim_{h\to 0}\frac{5^h-1}{h}\approx 1.6094 \qquad \lim_{h\to 0}\frac{0.20^h-1}{h}\approx -1.6094$$

$$\lim_{h\to 0}\frac{9^h-1}{h}\approx 2.1972 \qquad \lim_{h\to 0}\frac{0.1111^h-1}{h}\approx -2.1973$$

Simply looking at the approximations to these limiting processes does not help us identify a pattern relating all of the values. However, that changes when we plot the ordered pairs

$$\left(B, \lim_{h\to 0}\frac{B^h-1}{h}\right)$$

for the various values of B we have considered.



These ordered pairs all lie on the curve that defines the function $y = \ln(B)$. This observation provides strong evidence for the following specific derivative formula.

Exponential Function Rule for Derivatives

If B is a positive constant, then the derivative function for the function $y = f(x) = B^x$ is defined by the formula

$$r = f'(x) = B^x \cdot \ln(B)$$

Problem 5. What is the point-slope formula for the line tangent to the graph of $y = f(x) = \left(\frac{1}{3}\right)^x$ at the point $(2, f(2))$?

Example 5. Differentiate the function $y = f(a) = a^2 + a^{-1}$.

Solution. Consider the average rate of change function

$$\begin{aligned} g_a(h) &= \frac{f(a+h) - f(a)}{h} \\ &= \left(\frac{1}{h}\right)\left(\left[(a+h)^2 + \frac{1}{a+h}\right] - \left[a^2 + \frac{1}{a}\right]\right) \\ &= \left(\frac{1}{h}\right)\left([(a+h)^2 - a^2] + \left[\frac{1}{a+h} - \frac{1}{a}\right]\right) \\ &= \left(\frac{1}{h}\right)[(a+h)^2 - a^2] + \left(\frac{1}{h}\right)\left[\frac{1}{a+h} - \frac{1}{a}\right] \end{aligned}$$

By simply rearranging the terms in the formula for the function g_a we have recast the function as the sum of two average rate of change functions that we have already worked with. In particular, we can see that

$$\begin{aligned} \frac{d}{da}[a^2 + a^{-1}] &= \lim_{h \to 0}\left\{\left(\frac{1}{h}\right)[(a+h)^2 - a^2] + \left(\frac{1}{h}\right)\left[\frac{1}{a+h} - \frac{1}{a}\right]\right\} \\ &= \lim_{h \to 0}\left(\frac{1}{h}\right)[(a+h)^2 - a^2] + \lim_{h \to 0}\left(\frac{1}{h}\right)\left[\frac{1}{a+h} - \frac{1}{a}\right] \\ &= \frac{d}{da}[a^2] + \frac{d}{da}[a^{-1}] \\ &= 2a - \frac{1}{a^2} \end{aligned}$$

****

In the previous example, we were tasked with constructing the derivative function for a function f whose formula was the sum of two functions whose derivatives we have already determined; and it turned out that the derivative function for f is simply the sum of the derivative formulas for the two component functions. This is a special example of the following rule.

Sum of Functions Rule for Derivatives

If $y = f(x)$ and $y = g(x)$ are differentiable functions, then the sum of these functions is also differentiable. In fact,

$$\frac{d}{dx}[f(x) + g(x)] = \frac{d}{dx}[f(x)] + \frac{d}{dx}[g(x)]$$

Notice that this rule does not provide the formula for the derivative function of a *specific* function. Instead, it tells us how to determine the derivative for a sum of functions if we know the derivative for each function individually. The Sum Rule is an example of a *general derivative rule*.

Example 6. Suppose $y = f(x) = x^3 + x^{-1} - 10$. At what input values will we have $f'(x) = 2$?

Solution. We want to solve the equation $f'(x) = 2$ for the unknown x. First, observe that

$$\begin{aligned} \frac{d}{dx}[x^3 + x^{-1} - 10\,] &= \frac{d}{dx}[x^3] + \frac{d}{dx}[x^{-1}] + \frac{d}{dx}[-10] && \text{Sum Rule for Derivatives} \\ &= 3x^2 - \frac{1}{x^2} + 0 && \text{Power \& Constant Function Rules} \end{aligned}$$

We therefore want to solve the equation

$$3x^2 - \frac{1}{x^2} = 2$$

for the unknown x. In order to solve this equation, we will need to rearrange it so that all expressions involving the unknown x appear in the numerator. Observe

$$3x^2 - \frac{1}{x^2} = 2 \quad \Rightarrow \quad \frac{3x^4 - 1}{x^2} = 2 \quad \Rightarrow \quad 3x^4 - 1 = 2x^2 \quad \Rightarrow \quad 3x^4 - 2x^2 - 1 = 0$$

If we make the variable substitution $u = x^2$, then the equation on the far right above becomes a quadratic with respect to the unknown u. In particular, we see that

$$\begin{aligned} 3x^4 - 2x^2 - 1 = 0 \quad &\Rightarrow \quad 3u^2 - 2u - 1 = 0 \\ &\Rightarrow \quad u = -\frac{(-2)}{2 \cdot 3} \pm \frac{\sqrt{(-2)^2 - 4(3)(-1)}}{2 \cdot 3} && \text{Apply Quadratic Formula} \\ &\Rightarrow \quad u = \frac{1}{3} \pm \frac{2}{3} \\ &\Rightarrow \quad u = -\frac{1}{3} \quad \text{or} \quad u = 1 \\ &\Rightarrow \quad x^2 = -\frac{1}{3} \quad \text{or} \quad x^2 = 1 \\ &\Rightarrow \quad x = \pm 1 && \text{(The equation } x^2 = -\tfrac{1}{3} \text{ has no solution.)} \end{aligned}$$

Consequently, the function $f(x) = x^3 + x^{-1} - 10$ will have an instantaneous rate of change of 2 when $x = 1$ and when $x = -1$.

****

Constant Multiple Rule for Derivatives

If $y = f(x)$ is a differentiable function, and if C is any constant, then the constant-multiple function defined by $y = g(x) = C \cdot f(x)$ is also differentiable. In fact,

$$\frac{d}{dx}[C \cdot f(x)] = C \cdot \frac{d}{dx}[f(x)]$$

The Constant Multiple Rule is another general rule for derivatives. To see why this general rule should be true, consider the average rate of change function

$$A_x(h) = \frac{g(x+h) - g(x)}{h} = \frac{C \cdot f(x+h) - C \cdot f(x)}{h} = C \cdot \frac{f(x+h) - f(x)}{h}$$

Since the function f is assumed to be differentiable, we may conclude

$$\frac{d}{dx}[g(x)] = C \cdot \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = C \cdot \frac{d}{dx}[f(x)]$$

Example 7. What is the formula for the derivative function of $= f(x) = 3\sqrt{x} + 5 \cdot 4^x + 8$?

Solution. Observe that

$$\frac{d}{dx}[3\sqrt{x} + 5 \cdot 4^x + 8\,] = \frac{d}{dx}[3\sqrt{x}] + \frac{d}{dx}[5 \cdot 4^x] + \frac{d}{dx}[8] \qquad \text{Sum Rule for Derivatives}$$

$$= 3\frac{d}{dx}[\sqrt{x}] + 5 \cdot \frac{d}{dx}[4^x] + \frac{d}{dx}[8] \qquad \text{Constant Multiple Rule for Derivatives}$$

$$= \frac{3}{2\sqrt{x}} + 4^x \ln(4) + 0 \qquad \text{Power Rule, Exponential Rule, \& Constant Function Rule}$$

(Note that we took for granted the fact that $\sqrt{x} = x^{1/2}$.)

****

Problem 6. If $y = f(t) = 2t^{2/3} - 6 \cdot e^t$, then what is the value of $f'(3)$? Remember, the symbol *e* represents *euler's constant*. ($e \approx 2.7183$)

HOMEWORK: Section 3.1 (Pages 180 – 181) Problems 3, 4, 5, 7, 9, 11, 13, 14, 15, 16, 17, 19, 21, 24, 32, 33, 34, 35, 43