In this discussion, we will explore general derivative rules that tell us how to differentiate products and quotients of differentiable functions. Let's start with a problem.

Problem 1. Let $y = f(x) = x^{12}$, and let $y = u(x) = x^4$ and $y = v(x) = x^8$.

Part (a): Use the power rule to differentiate the functions f, u, and v.

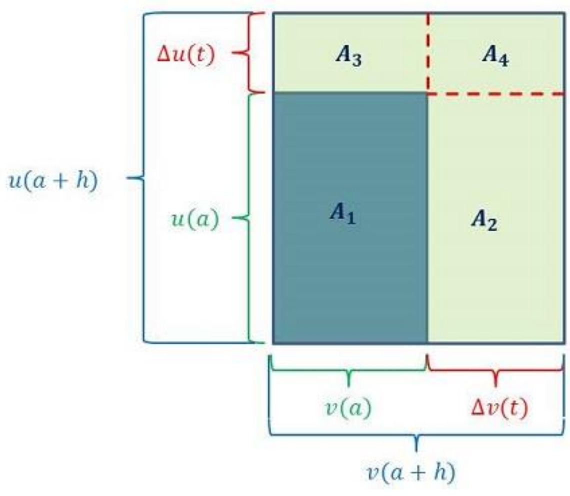
$$\frac{d}{dx}[x^{12}] = \qquad \frac{d}{dx}[x^4] = \qquad \frac{d}{dx}[x^8] =$$

Part (b): The laws of exponents tell us that $f(x) = u(x) \cdot v(x)$; consequently, there ought to be a relationship between $f'(x)$ and the functions $u'(x)$ and $v'(x)$. Show that it is possible to write the function f' as a combination of the four functions u, v, u', and v'.

Suppose we have a rectangle whose length and width can change with time. Let $L = u(t)$ denote the length in inches of the rectangle as a function of the number of minutes t since the dimensions of the rectangle began changing. Let $W = v(t)$ denote the width in inches of the rectangle as a function of the number of minutes t since the dimensions of the rectangle began changing. The area A of the rectangle is also a function of the variable t. In particular, we know

$$A = f(t) = u(t) \cdot v(t)$$

The diagram below shows two snapshots of the rectangle, one taken at time $t = a$ minutes, and one taken at $t = a + h$ minutes. The two snapshots have been superimposed.



In the diagram, the area of the rectangle has increased from $A_1 = u(a) \cdot v(a)$ to the area

$$A_1 + A_2 + A_3 + A_4 = u(a+h) \cdot v(a+h)$$

Consequently, the average rate of change for the area of the rectangle with respect to time on the interval from $t = a$ to $t = a + h$ minutes is given by

$$\frac{u(a+h)\cdot v(a+h) - u(a)\cdot v(a)}{h} = \frac{[A_1 + A_2 + A_3 + A_4] - A_1}{h} = \frac{A_2 + A_3 + A_4}{h}$$

Now, observe that

$$A_2 = u(a)\cdot \Delta v(t) = u(a)(v(a+h) - v(a))$$

$$A_3 = v(a)\cdot \Delta u(t) = v(a)(u(a+h) - u(a))$$

$$A_4 = \Delta v(t)\cdot \Delta u(t) = \ v(a)(u(a+h) - u(a))\cdot (u(a+h) - u(a))$$

Therefore, we know that

$$\frac{u(a+h)\cdot v(a+h) - u(a)\cdot v(a)}{h} = u(a)\cdot \frac{v(a+h) - v(a)}{h} + v(a)\frac{u(a+h) - u(a)}{h} + \frac{\Delta v(t)\cdot \Delta u(t)}{h}$$

Now, if we assume that the functions u and v are both differentiable, we know that

$$\lim_{h\to 0}\frac{v(a+h) - v(a)}{h} = v'(a) \qquad\qquad \lim_{h\to 0}\frac{u(a+h) - u(a)}{h} = u'(a)$$

Consequently, we also know that

$$f'(a) = \lim_{h\to 0}\frac{u(a+h)\cdot v(a+h) - u(a)\cdot v(a)}{h}$$

$$= u(a)\cdot v'(a) + v(a)\cdot u'(a) + \lim_{h\to 0}\frac{\Delta v(t)\cdot \Delta u(t)}{h}$$

$$= u(a)\cdot v'(a) + v(a)\cdot u'(a) + \lim_{h\to 0}\left[\big(v(a+h) - v(a)\big)\cdot \frac{u(a+h) - u(a)}{h}\right]$$

$$= u(a)\cdot v'(a) + v(a)\cdot u'(a) + 0\cdot u'(a)$$

$$= u(a)\cdot v'(a) + v(a)\cdot u'(a)$$

Product Rule for Derivatives

If $y = u(x)$ and $y = v(x)$ are differentiable functions, then the function $y = f(x) = u(x)\cdot v(x)$ is also differentiable, and

$$f'(x) = u(x)\cdot v'(x) + v(x)\cdot u'(x)$$

Example 1. Use the specific derivative formulas and the product rule to differentiate the function

$$y = 3x^{-3/4} \cdot 4^x$$

Solution. When using derivative rules to differentiate a function, the general strategy is always the same. You apply general rules to break up the formula for the function into combinations of derivatives of progressively simpler formulas, until you are faced with derivatives that are *exact matches* to the specific derivative rules.

The formula is a product of two functions

$$\frac{dy}{dx} = \frac{d}{dx}\big[\overbrace{\underbrace{3x^{-3/4}}_{u(x)} \cdot \underbrace{4^x}_{v(x)}}\big]$$

$$= 3\,x^{-3/4} \cdot \frac{d}{dx}[4^x] + 4^x \cdot \frac{d}{dx}[3\,x^{-3/4}] \qquad \textit{Apply Product Rule}$$

$$= 3\,x^{-3/4} \cdot \frac{d}{dx}[4^x] + 4^x \cdot 3\,\frac{d}{dx}[\,x^{-3/4}] \qquad \textit{Apply Constant Multiple Rule}$$

$$= 3\,x^{-3/4} \cdot 4^x \ln(4) + 4^x \cdot 3\left(-\frac{3}{4}x^{-7/4}\right) \qquad \textit{Apply Specific Formulas}$$

$$= 3\,x^{-3/4} \cdot 4^x \ln(4) - \frac{9}{4}x^{-7/4} \cdot 4^x \qquad \textit{Make easy simplifications}$$

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Problem 2. Differentiate the function $y = f(x) = 4x^{-1} \cdot \sqrt[3]{x}$.

Problem 3. What is the instantaneous rate of change for the function $a = h(t) = 4t - t^2 e^t$ at the input value $t = 1$?

Quotient Rule for Derivatives

Suppose that $y = U(x)$ and $y = L(x)$ are differentiable functions. As long as $L(x) \neq 0$, the function

$$y = f(x) = \frac{U(x)}{L(x)}$$

is also differentiable, and

$$f'(x) = \frac{L(x) \cdot U'(x) - U(x) \cdot L'(x)}{L(x) \cdot L(x)}$$

The quotient rule for derivatives is certainly not a very intuitive result, at least on first reading. However, the motivation behind the formula becomes a little clearer if we observe that on the input interval from $x = a$ to $x = a + h$, we have

$$f(a+h) - f(a) = \frac{U(a+h)}{L(a+h)} - \frac{U(a)}{L(a)}$$

$$= \frac{L(a) \cdot U(a+h) - U(a) \cdot L(a+h)}{L(a) \cdot L(a+h)}$$

$$= \frac{L(a) \cdot U(a+h) + [\boldsymbol{L(a) \cdot U(a) - L(a) \cdot U(a)}] - U(a) \cdot L(a+h)}{L(a) \cdot L(a+h)}$$

$$= \frac{L(a)[U(a+h) - U(a)] - U(a)[L(a+h) - L(a)]}{L(a) \cdot L(a+h)}$$

Consequently, we know that

$$\frac{f(a+h) - f(a)}{h} = \left(\frac{1}{h}\right)\left\{\frac{L(a)[U(a+h) - U(a)] - U(a)[L(a+h) - L(a)]}{L(a) \cdot L(a+h)}\right\}$$

$$= \frac{1}{L(a) \cdot L(a+h)}\left(L(a)\frac{U(a+h) - U(a)}{h} - U(a)\frac{L(a+h) - L(a)}{h}\right)$$

Since we have assumed that the functions U and L are differentiable, we know that

$$\lim_{h \to 0} \frac{U(a+h) - U(a)}{h} = U'(a) \qquad \lim_{h \to 0} \frac{L(a+h) - L(a)}{h} = L'(a)$$

Therefore, we also know that

$$f'(a) = \lim_{h \to 0} \frac{1}{L(a) \cdot L(a+h)}\left(L(a)\frac{U(a+h) - U(a)}{h} - U(a)\frac{L(a+h) - L(a)}{h}\right)$$

$$= \frac{1}{L(a) \cdot L(a)}\big(L(a)U'(a) - U(a)L'(a)\big)$$

Example 2. What is the formula for the tangent line to the graph of the function

$$p = q(r) = 4r - \frac{r}{1+r^2}$$

at the point $(2, q(2))$?

Solution. We know that the point-slope formula for the tangent line to the graph will have the form $p = q'(2)[r-2] + q(2)$. We need the values of $q(2)$ and $q'(2)$. Now,

$$q(2) = 4(2) - \frac{1}{1+(2)^2} = \frac{39}{5}$$

$$\frac{dp}{dr} = \frac{d}{dr}\left[\overbrace{4r - \frac{r}{1+r^2}}^{\text{This formula is a sum of two functions}}\right]$$

$$= 4\frac{d}{dr}[r] \; - \; \frac{d}{dr}\overbrace{\left[\frac{r}{1+r^2}\right]}^{\text{This formula is a quotient of two functions}}$$ Apply Sum and Constant Multiple Rules

$$= 4\frac{d}{dr}[r] - \left[\frac{1}{(1+r^2)^2}\right]\left[(1+r^2)\frac{d}{dr}[r] - r\frac{d}{dr}\overbrace{[1+r^2]}^{\text{This formula is a sum of two functions}}\right]$$ Apply Quotient Rule

$$= 4\frac{d}{dr}[r] - \left[\frac{1}{(1+r^2)^2}\right]\left[(1+r^2)\frac{d}{dr}[r] - r\left(\frac{d}{dr}[1] + \frac{d}{dr}[r^2]\right)\right]$$ Apply Sum Rule

$$= 4(1) - \left[\frac{1}{(1+r^2)^2}\right][(1+r^2)(1) - r(0+2r)]$$ Apply specific derivative formulas

$$= 4 - \frac{1-r^2}{(1+r^2)^2}$$

Now that we have the formula for the function q', we can see that

$$q'(1) = 4 - \frac{1-(2)^2}{(1+(2)^2)^2} = \frac{97}{25}$$

The formula for the tangent line will therefore be $p = \frac{97}{25}[r-2] + \frac{39}{5}$.

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Problem 4. Differentiate the function $s = y(t) = \frac{t^2}{1-t^2}$.

Problem 5. What is the formula for the tangent line to the graph of the function

$$m = z(n) = e^n - \frac{2n}{1-n}$$

at the point $(0, z(0))$?

HOMEWORK: Section 3.2 (Page 188) Problems 3, 4, 5, 7, 9, 12, 13, 14, 15, 19, 20, 21