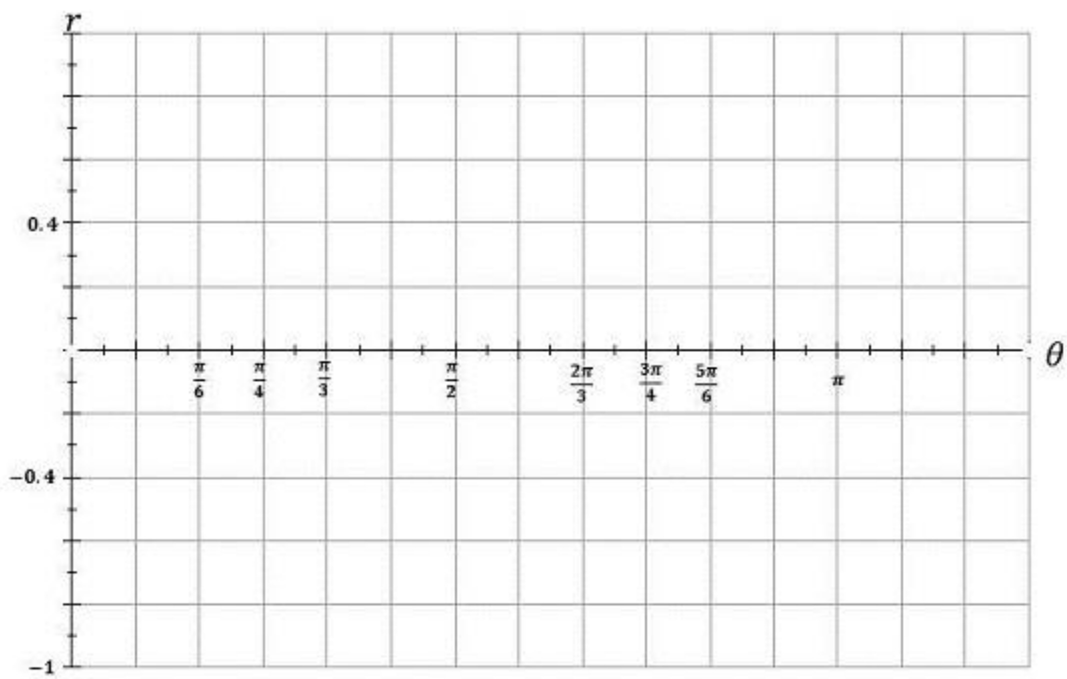
In this discussion, we will develop the special derivative formulas for the sine, cosine, and tangent functions. Like the exponential functions, the trigonometric functions are transcendental. Because of this fact, working with limits that involve the trigonometric functions pose special challenges.

Let's begin by sketching the graph of the derivative function for the function $y = f(\theta) = \sin(\theta)$. For any input value $\theta = a$, we know that

$$f'(a) \approx \frac{\sin(a + 0.0001) - \sin(a)}{0.0001}$$

Make sure your calculator is in radian mode and use this formula to fill in the table below. Then, use the data in this table to sketch the graph of the function $r = f'(\theta)$ on the grid provided.

Value of θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π
Approximate value of $f'(\theta)$									



Does the graph of the derivative function look familiar to you?

The sketch of the graph for the derivative function suggests that $f'(\theta) = \cos(\theta)$. We can give more evidence for this being the case by looking more closely at the average rate of change function for $y = f(\theta) = \sin(\theta)$. For any input value $\theta = a$, the average rate of change for the sine function on the input interval from $\theta = a$ to $\theta = a + h$ is given by

$$g_a(h) = \frac{f(a+h) - f(a)}{h}$$

$$= \frac{\sin(a+h) - \sin(a)}{h}$$

$$= \frac{\sin(a)\cos(h) + \cos(a)\sin(h) - \sin(a)}{h}$$

Apply the sum formula for the sine function

$$= \sin(a) \cdot \left[\frac{\cos(h) - 1}{h}\right] + \cos(a) \cdot \left[\frac{\sin(h)}{h}\right]$$

Consequently, we know that

$$f'(a) = \sin(a) \lim_{h \to 0} \frac{\cos(h) - 1}{h} + \cos(a) \lim_{h \to 0} \frac{\sin(h)}{h}$$

We need to know the values of the two limiting processes that appear in the formula above. Unfortunately, neither of these limit processes can be computed using basic algebra. However, we can estimate the values of these limiting processes by choosing very small values of h. For example,

$$\lim_{h \to 0} \frac{\cos(h) - 1}{h} \approx \frac{\cos(0.00001) - 1}{0.00001} \approx -0.000005$$

$$\lim_{h \to 0} \frac{\sin(h)}{h} \approx \frac{\sin(0.00001)}{0.00001} \approx 1$$

From these estimates, it seems safe to say that

$$\lim_{h \to 0} \frac{\cos(h) - 1}{h} = 0 \qquad \lim_{h \to 0} \frac{\sin(h)}{h} = 1$$

It is possible to prove that these equations are correct; however, verification requires a rather lengthy foray into trigonometry and the formal properties of limits.

Problem 1. If $y = f(t) = t^2 \sin(t)$, then what is the value of $f'(2)$?

We can use a similar approach to determine the formula for the derivative function of the cosine function. If we let $y = f(\theta) = \cos(\theta)$, then for any input value $\theta = a$, the average rate of change for the cosine function on the input interval from $\theta = a$ to $\theta = a + h$ is given by

$$g_a(h) = \frac{f(a+h) - f(a)}{h}$$

$$= \frac{\cos(a+h) - \cos(a)}{h}$$

$$= \frac{\cos(a)\cos(h) - \sin(a)\sin(h) - \cos(a)}{h}$$

Apply the sum formula for the cosine function

$$= \cos(a) \cdot \left[\frac{\cos(h) - 1}{h}\right] - \sin(a) \cdot \left[\frac{\sin(h)}{h}\right]$$

Consequently, we know that

$$f'(a) = \cos(a) \lim_{h\to 0} \frac{\cos(h) - 1}{h} - \sin(a) \lim_{h\to 0} \frac{\sin(h)}{h} = -\sin(a)$$

Problem 2. The *secant* function is defined by the formula

$$y = \sec(\theta) = \frac{1}{\cos(\theta)}$$

Use the quotient rule to determine a formula for the derivative function for the secant function.

Problem 3. The *tangent* function is defined by the formula

$$y = \tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)}$$

Use the quotient rule and the *Pythagorean Identity* $[\sin(\theta)]^2 + [\cos(\theta)]^2 = 1$ to determine a formula for the derivative function for the tangent function.

Specific Derivative Formulas

1. If $y = f(x) = a$ is a constant function, then $f'(x) = 0$.
2. If r is a rational number and $y = f(x) = x^r$, then $f'(x) = rx^{r-1}$.
3. If $y = f(x) = a^x$ for any positive constant a, then $f'(x) = a^x \ln(a)$.
4. If $y = f(x) = \sin(x)$, then $f'(x) = \cos(x)$.
5. If $y = f(x) = \cos(x)$, then $f'(x) = -\sin(x)$.
6. If $y = f(x) = \sec(x)$, then $f'(x) = \sec(\mathrm{x}) \tan(x)$.
7. If $y = f(x) = \tan(x)$, then $f'(x) = [\sec(x)]^2$.

General Derivative Rules

Constant Multiple Rule
If $y = f(x)$ is a differentiable function and K is any constant, then $\frac{d}{dx}[Kf(x)] = K\frac{d}{dx}[f(x)]$.

Sum Rule
If $y = f(x)$ and $y = g(x)$ are differentiable functions, then $\frac{d}{dx}[f(x) + g(x)] = \frac{d}{dx}[f(x)] + \frac{d}{dx}[g(x)]$.

Product Rule
If $y = f(x)$ and $y = g(x)$ are differentiable functions, then

$$\frac{d}{dx}[f(x) \cdot g(x)] = f(x) \cdot \frac{d}{dx}[g(x)] + g(x)\frac{d}{dx}[f(x)]$$

Quotient Rule
If $y = f(x)$ and $y = g(x)$ are differentiable functions, then

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{1}{g(x) \cdot g(x)}\left[g(x) \cdot \frac{d}{dx}[f(x)] - f(x) \cdot \frac{d}{dx}[g(x)]\right]$$