Suppose that $y = f(u)$ and $u = g(x)$ are functions, and suppose that the output values for the function g are all members of the domain for the function f (so that the composite function $f \circ g$ is defined on the domain of the function g).

In the last investigation, we provided some evidence that there is a simple relationship between the following quantities:

- the average rate of change for the composite function $f \circ g$ with respect to x on the input interval from $x = a$ to $x = a + h$
- the average rate of change for the function g with respect to x on the input interval $x = a$ to $x = a + h$
- the average rate of change for the function f with respect to u on the input interval $u = g(a)$ to $u = g(a + h)$

In particular, we provided evidence that

$$\frac{\triangle f(g(x))}{\triangle x} = \frac{\triangle f(u)}{\triangle u} \cdot \frac{\triangle g(x)}{\triangle x}$$

Basic algebra shows why this result should be true in general. Observe that on the input interval from $x = a$ to $x = a + h$, we have

$$\begin{aligned}\frac{\triangle f(g(x))}{\triangle x} &= \frac{[f \circ g](a+h) - [f \circ g](a)}{h} \\ &= \left(\frac{f\big(g(a+h)\big) - f(g(a))}{g(a+h) - g(a)}\right) \cdot \left(\frac{g(a+h) - g(a)}{h}\right) \\ &= \frac{\triangle f(u)}{\triangle u} \cdot \frac{\triangle g(x)}{\triangle x}\end{aligned}$$

(At least as long as $g(a + h) - g(a) \neq 0.$) The issue with division by 0 notwithstanding, this derivation suggests the following result.

The Chain Rule for Derivatives

If $y = f(u)$ and $u = g(x)$ are differentiable functions of the input variables u and x, respectively, then the composite function $y = [f \circ g](x)$ is a differentiable function of the input variable x. Furthermore,

$$\frac{d}{dx}[f(g(x))] = \frac{dg}{dx} \cdot \left.\frac{df}{du}\right|_{u=g(x)} = g'(x) \cdot f'(g(x))$$

Example 1. Use the chain rule to differentiate the function $y = h(x) = \sqrt{x + \sin(x)}$.

Solution. The function h is not an exact match to any of the special derivative formulas we have encountered thus far, so we know that we must start with one of the general derivative rules. Now, the function h can be thought of as the composition of two differentiable functions:

$$u = g(x) = x + \sin(x) \qquad \text{and} \qquad y = f(u) = \sqrt{u}$$

$$\frac{dg}{dx} = \frac{d}{dx}[x + \sin(x)] = 1 + \cos(x)$$

$$\frac{df}{du} = \frac{d}{du}\left[\sqrt{u}\right] = \frac{1}{2\sqrt{u}}$$

Consequently, the Chain Rule for derivatives tells us

$$\frac{dh}{dx} = \frac{dg}{dx} \cdot \left.\frac{df}{du}\right|_{u=g(x)} = (1 + \cos(x)) \cdot \frac{1}{2\sqrt{x + \sin(x)}} = \frac{1 + \cos(x)}{2\sqrt{x + \sin(x)}}$$

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Problem 1. Consider the function $y = h(x) = \cos(x^{-4})$.

Part (a): Determine functions $u = g(x)$ and $y = f(u)$ such that $h(x) = f(g(x))$.

$$\frac{dg}{dx} = \qquad\qquad\qquad\qquad \frac{df}{du} =$$

$$\frac{dh}{dx} =$$

Problem 2. Consider the function $y = h(x) = [x^2 + 3\sin(x)]^4$.

Part (a): Determine functions $u = g(x)$ and $y = f(u)$ such that $h(x) = f(g(x))$.

$$\frac{dg}{dx} = \qquad\qquad\qquad\qquad \frac{df}{du} =$$

$$\frac{dh}{dx} =$$

Problem 3. Use the Product Rule and the Chain Rule to differentiate the function

$$y = h(x) = \sin(x)\tan(x^2)$$

Example 2. Differentiate the function $y = j(x) = \sin^3(x^{-1})$.

Solution. It is a (somewhat unfortunate) custom to write $\sin^3(\theta)$ as a shorthand for $[\sin(\theta)]^3$. With this in mind, the function j is actually a composition of three functions.

$$j(x) = \sin^3(x^{-1}) = \Big[\ \overbrace{\underbrace{\sin(\overbrace{x^{-1}}^{v=g(x)})}_{u=f(v)}}^{y=h(u)}\ \Big]^3$$

$$v = g(x) = x^{-1} \qquad u = f(v) = \sin(v) \qquad y = h(u) = u^3$$

$$\frac{dg}{dx} = -\frac{1}{x^2} \qquad \frac{df}{dv} = \cos(v) \qquad \frac{dh}{du} = 3u^2$$

Note that $j(x) = h(f(g(x)))$. Now, the Chain Rule tells us

$$\frac{dj}{dx} = \frac{dg}{dx}\cdot\left(\frac{df}{dv}\bigg|_{v=g(x)}\right)\cdot\left(\frac{dh}{du}\bigg|_{u=f(g(x))}\right)$$

$$= -\frac{1}{x^2}\cdot\cos(x^{-1})\cdot 3[\sin(x^{-1})]^2$$

$$= -\frac{3\cos(x^{-1})\sin^2(x^{-1})}{x^2}$$

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Problem 4. Differentiate the function $y = j(x) = \cos^2(e^x)$.

HOMEWORK: Section 3.4 (Page 204) Problems 7, 9, 10, 11, 12, 13, 14, 16, 21, 23