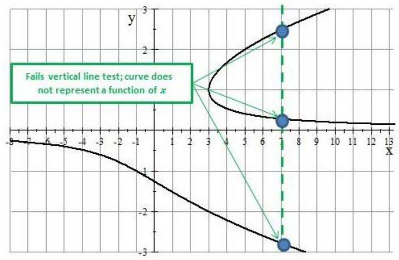
A relationship between two changing quantities represented by the variables x and y is said to be *implicit with respect to x and y* provided it is possible construct a formula $F(x,y)=a$ to represent the relationship, where a is a constant. We say that the relationship is *explicit with respect to x* (or *with respect to y*) when the formula may be recast as a function of either x or y.

Example 1. The formula $xy - y^3 = 2$ defines an implicit relationship with respect to the variables x and y.



The graph above shows that this relationship cannot be expressed as a function of the variable x. However, the graph passes the horizontal line test; consequently it is possible to recast the relationship as a function of the variable y. We can do this simply by solving the equation for the variable x. Observe that

$$xy - y^3 = 2 \quad \Rightarrow \quad x = \frac{2+y^3}{y} \quad (y \neq 0)$$

We can therefore refine our classification and say that this relationship is *explicit* with respect to the variable y but *not explicit* with respect to the variable x.

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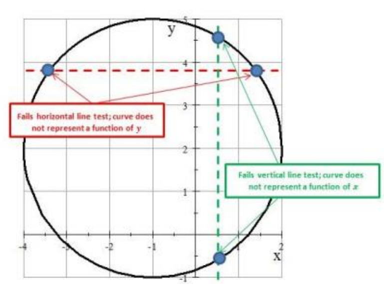
Example 2. The formula for a circle of radius 3 centered at the point $(-1,2)$ is

$$(x+1)^2 + (y-2)^2 = 9$$

The graph of the circle fails both the horizontal and vertical line tests. Consequently, the formula for this circle describes an implicit relationship with respect to x and y that cannot be made explicit with respect to either variable.



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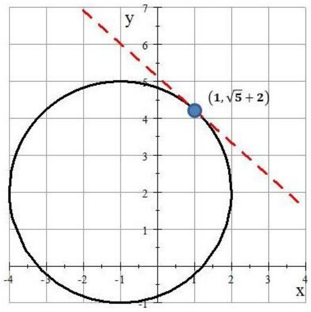
Most implicit relationships do not represent explicit functions of either variable; nevertheless, it is often possible to describe tangent lines to the graphs of these relationships. For example, we can draw the tangent line to the graph of the circle in Example 2 at any point.



In the diagram above, we have drawn the tangent line to the graph of the circle in Example 2 at the point $(1, \sqrt{5}+2)$. Based on the diagram, we see that the slope of this line is

$$\frac{\Delta y}{\Delta x} \approx \frac{\left(\sqrt{5}+2\right) - 6.90}{1-(-2)} \approx -0.888$$

Therefore, the point-slope formula for this tangent line is approximately

$$y = -0.888(x-1) + \sqrt{5} + 2$$

Even though the formula defining the circle is not a function of either x or y, it is still possible to use the tools of calculus to determine the slope of the tangent line. The method we use is really just the Chain Rule, written in a slightly different way.

Implicit Differentiation
If $y = f(x)$ is a differentiable function of x, then the derivative of f with respect to the variable u is given by $$\frac{df}{du} = \frac{df}{dx} \cdot \frac{dx}{du}$$

Suppose we want to differentiate an implicit relationship $F(x, y) = a$ with respect to the variable x. We do so by applying the rules for differentiation in the usual way; however, every time we encounter expressions involving the variable y, we treat each occurrence of y as an *unknown* (but differentiable) function of x and apply the Chain Rule.

Example 3. Find a formula for $\frac{dy}{dx}$ if x and y are related by the formula $xy = 1$.

Solution. We have an implicit formula relating x and y. It is possible to recast this formula as a function of the variable x simply by solving for y. Doing so, we see that

$$y = f(x) = \frac{1}{x}$$

Therefore, it is actually easy to determine the formula for $\frac{dy}{dx}$ using methods we are already familiar with. In particular,

$$\frac{dy}{dx} = f'(x) = -\frac{1}{x^2}$$

Now, suppose we did not recognize that the formula can be recast as a function of the variable x. We can still construct a formula for $\frac{dy}{dx}$.

$$xy = 1 \quad \Rightarrow \quad \frac{d}{dx}[xy] = \frac{d}{dx}[1]$$

Differentiate both sides of equation with respect to x.

Treat y as an unspecified function of x and apply the Product Rule.

$$\Rightarrow \quad \frac{d}{dx}[x] \cdot y + x \cdot \frac{d}{dx}[y] = \frac{d}{dx}[1]$$

We don't know how y depends on x so we cannot compute $\frac{dy}{dx}$.

$$\Rightarrow \quad (1) \cdot y + x \cdot \frac{dy}{dx} = 0$$

Apply specific derivative formulas where possible.

$$\Rightarrow \quad \frac{dy}{dx} = -\frac{y}{x}$$

Solve for the derivative $\frac{dy}{dx}$.

At first, it seems like we have two different formulas for $\frac{dy}{dx}$; however, if we now use the fact that

$$y = f(x) = \frac{1}{x}$$

notice that we can write

$$\frac{dy}{dx} = -\frac{y}{x} \quad \Rightarrow \quad \frac{dy}{dx} = -\frac{f(x)}{x} \quad \Rightarrow \quad \frac{dy}{dx} = -\frac{1}{x^2}$$

Consequently, the two formulas are really equivalent.

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Example 4. Find a formula for $\frac{dy}{dx}$ if x and y are related by the formula $2x - \cos(y) = 1$.

Solution. In this case, there is no easy way to recast the formula as a function of the variable x.

$$2x - \cos(y) = 1 \quad \Rightarrow \quad \frac{d}{dx}[2x - \cos(y)] = \frac{d}{dx}[1]$$

Differentiate both sides of equation with respect to x

$$\Rightarrow \quad 2\frac{d}{dx}[x] - \frac{d}{dx}[\cos(y)] = \frac{d}{dx}[1]$$

Apply Sum and Constant Multiple Rules

$$\Rightarrow \quad 2\frac{d}{dx}[x] - \frac{d}{dx}[y] \cdot \frac{d}{du}[\cos(u)]\Big|_{u=y} = \frac{d}{dx}[1]$$

Apply Chain Rule, treating y as unknown function of x

$$\Rightarrow \quad 2 + \sin(y) \cdot \frac{dy}{dx} = 0$$

Apply special derivative formulas

We cannot evaluate the operation $\frac{dy}{dx}$, because we do not know how y depends on x. Instead, we simply treat the operation as an unknown and solve for it:

$$2 + \sin(y) \cdot \frac{dy}{dx} = 0 \quad \Rightarrow \quad \frac{dy}{dx} = -\frac{2}{\sin(y)}$$

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Problem 1. Find a formula for $\frac{dy}{dx}$ if the variables x and y are related by the formula $3x^2 - y^3 = 4$.

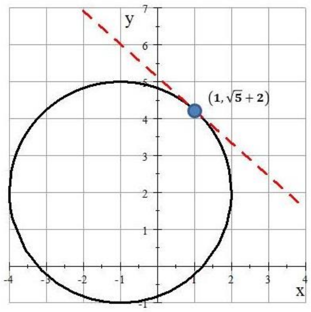
Problem 2. Find a formula for $\frac{du}{dv}$ if the variables u and v are related by the formula $uv^2 = 4\tan(u)$.

Problem 3. Find a formula for $\frac{da}{db}$ if the variables a and b are related by the formula $\cos(a) = (b^2 - 1)^3$.

Example 5. The circle of radius 3 centered at the point $(-1,2)$ is defined by the formula

$$(x+1)^2 + (y-2)^2 = 9$$

What is the point-slope formula for the line tangent to this circle at the point $(1, \sqrt{5}+2)$?



Solution. The slope of the this tangent line is given by the expression

$$\left.\frac{dy}{dx}\right|_{(x,y)=(1,\sqrt{5}+2)}$$

We therefore need to determine a formula for $\frac{dy}{dx}$. Observe that

$$9 = (x+1)^2 + (y-2)^2 \quad \Rightarrow \quad \frac{d}{dx}[9] = \frac{d}{dx}\left[(x+1)^2 + (y-2)^2\right]$$

$$\Rightarrow \quad \frac{d}{dx}[9] = \frac{d}{dx}\left[(x+1)^2\right] + \frac{d}{dx}\left[(y-2)^2\right]$$

$$\Rightarrow \quad \frac{d}{dx}[9] = \frac{d}{dx}[x+1] \cdot \left.\frac{d}{du}[u^2]\right|_{u=x+1} + \frac{d}{dx}[y-2] \cdot \left.\frac{d}{du}[u^2]\right|_{u=y-2}$$

$$\Rightarrow \quad \frac{d}{dx}[9] = \left(\frac{d}{dx}[x] + \frac{d}{dx}[1]\right) \cdot \left.\frac{d}{du}[u^2]\right|_{u=x+1} + \left(\frac{d}{dx}[y] - \frac{d}{dx}[2]\right) \cdot \left.\frac{d}{du}[u^2]\right|_{u=y-2}$$

$$\Rightarrow \quad 0 = (1+0) \cdot 2(x+1) + \left(\frac{dy}{dx} + 0\right) \cdot 2(y-1)$$

$$\Rightarrow \quad 0 = 2(x+1) + 2(y-1) \cdot \frac{dy}{dx}$$

$$\Rightarrow \quad -2(x+1) = 2(y-1) \cdot \frac{dy}{dx}$$

$$\Rightarrow -\frac{x+1}{y-2} = \frac{dy}{dx}$$

Consequently, the slope of the tangent line will be

$$\left.\frac{dy}{dx}\right|_{(x,y)=(1,\sqrt{5}+2)} = -\frac{1+1}{(\sqrt{5}+2)-2} = -\frac{2}{\sqrt{5}}$$

The point-slope formula for the tangent line is therefore given by

$$y = -\frac{2}{\sqrt{5}}(x-1) + \sqrt{5} + 2$$

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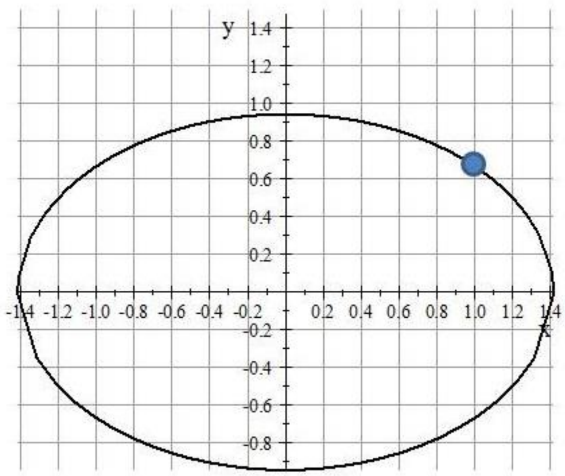
Problem 4. Consider the ellipse defined by the formula $4x^2 + 9y^2 = 8$. The graph of this ellipse is shown below.



Part (a): Carefully sketch the graph of the tangent line to the graph of the ellipse at the point $\left(1, \frac{2}{3}\right)$. Use your graph to estimate the slope of this tangent line.

Part (b): Find a formula for $\frac{dy}{dx}$ and use this formula to find the exact value of the slope for the tangent line at this point.

When we are working with a formula that relates two or more variables, we can differentiate that formula with respect to *any* variable we wish. We simply treat each variable in the formula as an unspecified function of the variable we are differentiating with respect to.

Example 6. Differentiate the function $b = x^2 + \cos(b)$ with respect to the variable y.

Solution. Observe that

$$b = x^2 + \cos(b) \quad \Rightarrow \quad \frac{db}{dy} = \frac{d}{dy}[x^2 + \cos(b)]$$

Apply the Sum Rule for derivatives

$$\Rightarrow \quad \frac{db}{dy} = \frac{d}{dy}[x^2] + \frac{d}{dy}[\cos(b)]$$

Treat x and b as unspecified functions of y and apply the Chain Rule.

$$\Rightarrow \quad \frac{db}{dy} = \frac{d}{dy}[x] \cdot \frac{d}{dw}[w^2]\bigg|_{w=x} + \frac{d}{dy}[b] \cdot \frac{d}{du}[\cos(u)]\bigg|_{u=b}$$

Apply specific derivative formulas where possible.

$$\Rightarrow \quad \frac{db}{dy} = \frac{dx}{dy} \cdot (2x) + \frac{db}{dy} \cdot (-\sin(b))$$

$$\Rightarrow \quad \frac{db}{dy} = 2x \cdot \frac{dx}{dy} - \sin(b) \cdot \frac{db}{dy}$$

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Problem 5. Differentiate the function $a = 4\tan(y) - 2^x$ with respect to the variable t.

Example 7. A drop of oil is placed on the surface of some water. The drop spreads across the surface as a circular disk. When the radius of the disk is two inches, and the radius is increasing that the instantaneous rate of 1.75 inches per second. At this moment, how fast is the area of the disk increasing with respect to the time since the oil drop was placed on the water?

Solution. There are three changing quantities in the problem --- the area of the disk, the radius of the disk, and the time since the oil drop was placed on the water.

- Let r represent the radius of the disk, measured in inches,
- Let t represent the time, measured in seconds, since the oil drop was placed on the water, and
- Let A represent the area of the disk, measured in square inches.

Now, we know that the values of both A and r covary with the values of t; however, we are not told *how* this covariance occurs. We do know how the values of A covary with the values of r. In particular, we know

$$A = f(r) = \pi r^2$$

Consequently, the Chain Rule tells us

$$\frac{dA}{dt} = \frac{d}{dt}[\pi r^2] \quad \Rightarrow \quad \frac{dA}{dt} = \frac{d}{dr}[\pi r^2] \cdot \frac{dr}{dt} \quad \Rightarrow \quad \frac{dA}{dt} = 2\pi r \cdot \frac{dr}{dt}$$

We are told that when $r = 2$ inches, the instantaneous rate of change for the radius is $\frac{dr}{dt} = 1.75$ inches per second. Consequently, at this instant, the instantaneous rate of change in the area of the disk will be

$$\frac{dA}{dt} = 2\pi(2 \text{ inches}) \cdot (1.75 \text{ inches per second}) \approx 19.028 \text{ square inches per second}$$

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Problem 6. A cube of ice is placed on a table, and is melting in such a way that it remains a cube. When the sides of the ice cube are three inches long, the length of each side is decreasing at the instantaneous rate of 0.8 inches per second. At this moment, what is the instantaneous rate of change in the volume of the cube with respect to the number of seconds since it was placed on the table?

HOMEWORK: Section 3.5 (Page 215) Problems 5, 7, 8, 10, 11, 25, 26, 27, 28