We can use the differentiation technique we applied to implicit formulas to obtain derivative formulas for a number of important functions.

For example, let's consider the logarithmic functions $y = f(x) = \log_a(x)$. The base-a logarithmic functions are defined to be the inverse functions for the base-a exponential functions. In particular, we know

$$y = \log_a(x) \quad \Rightarrow \quad x = a^y$$

In words, the value of $\log_a(x)$ is the power we must raise a to in order to obtain the number x. We can use this fact to obtain the formula for the derivative function of $y = f(x) = \log_a(x)$. Observe

$$\begin{aligned}
x = a^y \quad &\Rightarrow \quad \frac{d}{dx}[x] = \frac{d}{dx}[a^y] \\
&\Rightarrow \quad \frac{d}{dx}[x] = \frac{d}{dx}[y] \cdot \frac{d}{du}[a^u]\Big|_{u=y} \\
&\Rightarrow \quad 1 = \frac{dy}{dx} \cdot a^y \ln(a) \\
&\Rightarrow \quad \frac{1}{a^y \ln(a)} = \frac{dy}{dx} \\
&\Rightarrow \quad \frac{1}{x\ln(a)} = \frac{d}{dx}\left[\log_a(x)\right]
\end{aligned}$$

Derivative Formula for Logarithmic Functions

If a is any positive constant, then the derivative function for $y = f(x) = \log_a(x)$ is the function defined by

$$r = f'(x) = \frac{1}{x\ln(a)}$$

Example 1. What is the formula for the line tangent to the graph of $y = f(x) = \log_2(x)$ at the point $(8, f(8))$?

Solution. First, observe that $f(8) = 3$, since 3 is the power we must raise 2 to in order to obtain 8. Now,

$$f'(x) = \frac{d}{dx}\left[\log_2(x)\right] = \frac{1}{x\ln(2)}$$

Therefore, we know $f'(8) = \frac{1}{8\ln(8)}$. The formula for the tangent line is

$$y = \frac{1}{8\ln(8)}(x - 8) + 3$$

****

Example 2. What is the derivative function for the function $y = f(x) = x\ln(x) - x$?

Solution. Recall that $\ln(x) = \log_e(x)$, and also recall that $\ln(e) = 1$. With this in mind, observe

$$\begin{aligned}
f'(x) &= \frac{d}{dx}[x\ln(x) - x] \\
&= \frac{d}{dx}[x\ln(x)] - \frac{d}{dx}[x] \\
&= \frac{d}{dx}[x] \cdot \ln(x) + x \cdot \frac{d}{dx}[\ln(x)] - \frac{d}{dx}[x] \\
&= (1) \cdot \ln(x) + x \cdot \frac{1}{x\ln(e)} - 1 \\
&= \ln(x) + x \cdot \left(\frac{1}{x}\right) - 1 \\
&= \ln(x)
\end{aligned}$$

****

Problem 1. Differentiate the function $y = f(x) = \ln(\tan(x))$ with respect to x.

Problem 2. Let $y = f(x) = x^2 \log_2(x)$. Are there any values of x where $f'(x) = 0$?

A *sinusoid* is a function of the form

$$y = f(x) = A\sin[\omega(x-h)] + B$$

where A, B, ω, and H are constants. Sinusoid functions are widely used in the sciences and engineering to model one quantity that varies in an oscillating manner with respect to another quantity. Sinusoid functions are based on the basic sine function; and as such, are all transcendental functions. Consequently, it is impossible to solve an equation like

$$19 = 10\sin\left[\frac{\pi}{2}(x-1)\right] + 10$$

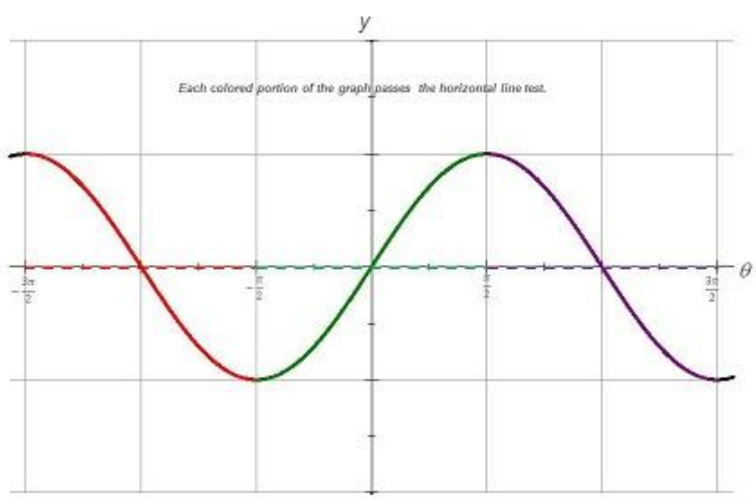
for the unknown x using only the tools of algebra. The best algebra can do for us is reduce the equation to

$$\frac{9}{10} = \sin\left[\frac{\pi}{2}(x-1)\right]$$

In order to "free" the unknown x from the input of the sine function, we need a function that *reverses* the sine function. In other words, we need a special function $u = g(y)$ with the property that $g(\sin(\theta)) = \theta$. If we had such a function, then we would know

$$\frac{9}{10} = \sin\left[\frac{\pi}{2}(x-1)\right] \quad \Rightarrow \quad g\left(\frac{9}{10}\right) = \frac{\pi}{2}(x-1)$$

We could now use the rules of algebra once again to isolate the unknown x. Unfortunately, the sine function is not one-to-one (its graph fails the horizontal line test). Consequently, no such function g exists --- at least not for the full domain of the sine function.



The diagram above shows the graph of the basic sine function $y = f(\theta) = \sin(\theta)$. Each colored portion represents an input interval where the graph passes the horizontal line test. On each of these colored intervals, it is possible to define a function that reverses the sine function *but only on that colored interval*. Each of these functions serves as a *partial inverse function* for the sine function.

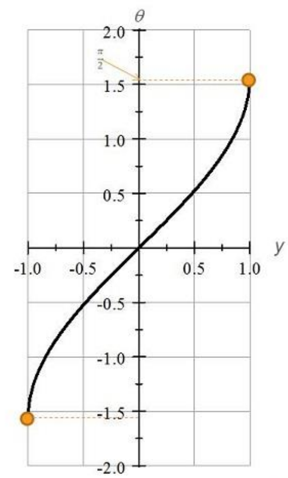
Principal Arcsine Function

The partial inverse function for the basic sine function defined on the input interval $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$ is called the *principal arcsine*. The principal arcsine function is denoted by $\theta = g(y) = \text{Arcsin}(y)$ and is defined by the relationship

$$\text{Whenever } -\frac{\pi}{2} \le \theta \le \frac{\pi}{2} \text{ we have } \theta = \text{Arcsin}(y) \text{ if and only if } y = \sin(\theta)$$

It is common to see the symbol " $\sin^{-1}$" used in place of the symbol "Arcsin" when working with the principal arcsine function. This is unfortunate, since it suggests that the sine function has a true inverse function.

The principal arcsine function is the only means available to tackle sinusoid equations when we are required to solve for values of the input variable. Here is a graph of the principal arcsine function.



Notice that the domain of the principal arcsine function is the interval $-1 \le y \le 1$. This limited domain reflects the fact that the *output* of the sine function is trapped between these numbers.

Example 3. Use the graph of the principal arcsine function to find one approximate solution to the equation

$$\frac{9}{10} = \sin\left[\frac{\pi}{2}(x-1)\right]$$

Solution. Since $\frac{9}{10} = 0.9$, the graph above tells us that $\text{Arcsin}\left(\frac{9}{10}\right) \approx 1.10$. Consequently, we know

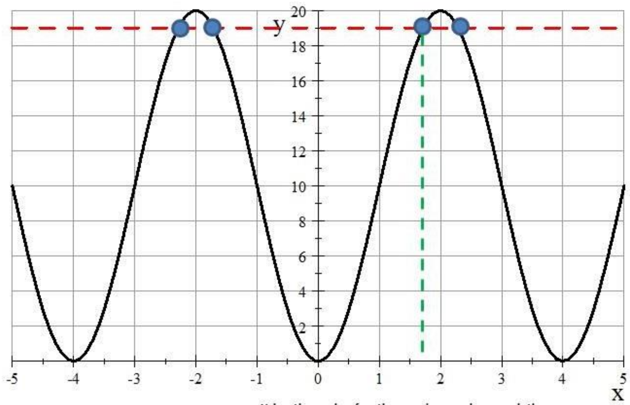
$$\frac{9}{10} = \sin\left[\frac{\pi}{2}(x-1)\right] \Rightarrow \text{Arcsin}\left[\frac{9}{10}\right] = \text{Arcsin}\left(\sin\left[\frac{\pi}{2}(x-1)\right]\right) \Rightarrow 1.10 \approx \frac{\pi}{2}(x-1)$$

Solving the last equation for the unknown x gives us the solution $x \approx 1.70$.

****

The approximate solution we obtained in Example 3 is not the whole story, however. There are, in fact, *infinitely many* solutions to the equation. To see why, the diagram below shows the graph of the functions

$$y = f(x) = 10\sin\left[\frac{\pi}{2}(x-1)\right] + 10 \quad \text{and} \quad y = 19$$



Using the arcsine function produces only one solution

The graph of the line $y = 19$ intersects the graph of the sinusoid infinitely many times. The x-coordinate of each intersection point is one solution to the equation.

It is possible to extract all of the other solutions to the equation from the one solution we obtained; however, the method by which this is accomplished is not the focus of this discussion. Instead, we will turn attention to developing a formula for the derivative function for $\theta = \text{Arcsin}(y)$.

We want to determine a formula for $\frac{d}{dy}[\text{Arcsin}(y).]$ Observe

$$\theta = \text{Arcsin}(y) \quad \Rightarrow \quad \sin(\theta) = y$$

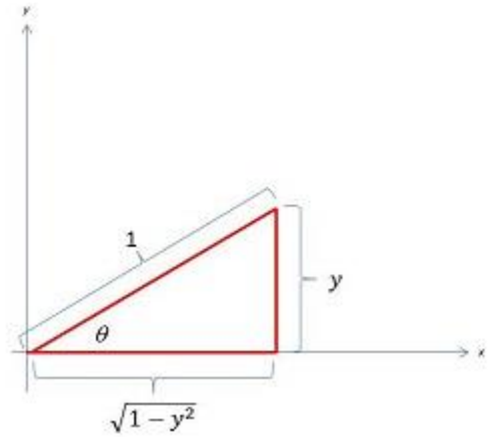
$$\Rightarrow \quad \frac{d}{dy}[\sin(\theta)] = \frac{d}{dy}[y]$$

$$\Rightarrow \quad \cos(\theta)\cdot\frac{d\theta}{dy} = 1$$

$$\Rightarrow \quad \frac{d\theta}{dy} = \frac{1}{\cos(\theta)}$$

$$\Rightarrow \quad \frac{d}{dy}[\text{Arcsin}(y)] = \sec(\text{Arcsin}(y))$$

It turns out that we can simplify this formula even further. The key to the simplification lies in the fact that, by construction, the output of the arcsine function *must* be an angle measure θ that lies between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$. This means that the angle with measure θ must be acute. It is therefore one angle in a right triangle.



In particular, we know that

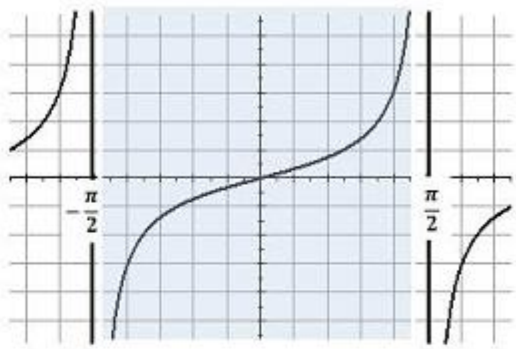
$$\frac{d}{dy}[\mathrm{Arcsin}(y)] = \sec\big(\mathrm{Arcsin}(y)\big) = \sec(\theta) = \frac{\text{Hypotenuse}}{\text{Side Adjacent } \theta} = \frac{1}{\sqrt{1-y^2}}$$

Problem 3. Differentiate the function $y = f(x) = \mathrm{Arcsin}(e^x)$ with respect to the variable x.

Problem 4. Differentiate the function $u = g(t) = \ln(t) \cdot \mathrm{Arcsin}(\cos(t))$ with respect to the variable t.

It is possible to define partial inverse functions for each of the trigonometric functions; however, only the partial inverse function for the tangent function is frequently encountered outside of mathematics.

Like the arcsine function, the *principal arctangent* function is defined by first selecting an input interval where the graph of the tangent function passes the horizontal line test. The diagram below shows the input interval that is used.

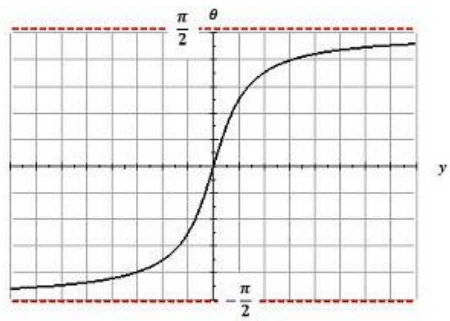


Input Interval for the Tangent Function
Used to Define the Principal Arctangent Function

Principal Arctangent Function

The partial inverse function for the basic tangent function defined on the input interval $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$ is called the *principal arctangent*. The principal arctangent function is denoted by $\theta = g(y) = \text{Arctan}(y)$ and is defined by the relationship

Whenever $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$ we have $\theta = \text{Arctan}(y)$ if and only if $y = \tan(\theta)$



Graph of the Principal Arctangent Function

We want to determine a formula for $\frac{d}{dy}[\text{Arctan}(y).]$ Observe

$$\theta = \text{Arctan}(y) \quad \Rightarrow \quad \tan(\theta) = y$$

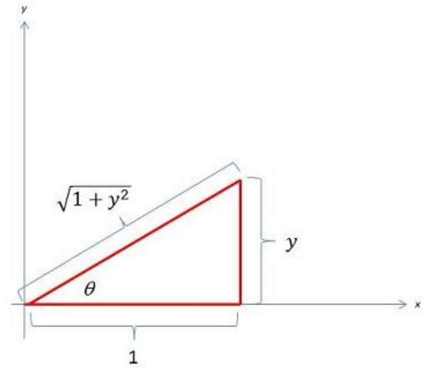
$$\Rightarrow \quad \frac{d}{dy}[\tan(\theta)] = \frac{d}{dy}[y]$$

$$\Rightarrow \quad \sec^2(\theta) \cdot \frac{d\theta}{dy} = 1$$

$$\Rightarrow \quad \frac{d\theta}{dy} = \frac{1}{\sec^2(\theta)}$$

$$\Rightarrow \quad \frac{d}{dy}[\text{Arctan}(y)] = \cos^2(\text{Arctan}(y))$$

It turns out that we can simplify this formula even further. The key to the simplification lies in the fact that, by construction, the output of the arctangent function *must* be an angle measure θ that lies between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$. This means that the angle with measure θ must be acute. It is therefore one angle in a right triangle.



In particular, we know that

$$\frac{d}{dy}[\text{Arctan}(y)] = \cos^2(\text{Arctan}(y)) = \cos^2(\theta) = \left(\frac{\text{Side Adjacent } \theta}{\text{Hypotenuse}}\right)^2 = \frac{1}{1+y^2}$$

Problem 5. At what values of x will the tangent line to the graph of $y = f(x) = \text{Arctan}(x - x^2)$ be horizontal?

Problem 6. Differentiate the function $a = f(b) = \text{Arctan}(\log_3 b)$ with respect to the variable b.

HOMEWORK: Section 3.5 (Page 216) Problems 49, 51, 56, 57
Section 3.6 (Page 223) Problems 2, 3, 5, 10, 11, 15