We have already seen that the derivative function encodes a considerable amount of information about the graph of the function it is created from. In particular, we have already worked with the following observations (albeit only with graphs).

First Derivative Test

Let $y = f(x)$ be a function and let $r = f'(x)$ represent its derivative function. Suppose that $f(a)$ exists.

- If the output of f' changes from positive to negative at $x = a$, then the function f has a local maximum output at $x = a$.
- If the output of f' changes from negative to positive at $x = a$, then the function f has a local minimum output at $x = a$.

(This test assumes we are reading the graph from left to right.)

The output of the derivative function can only change sign at a critical number for the function f. This suggests that we can apply the First Derivative Test directly to the derivative formula to identify local extrema for a function f.

Procedure for Applying First Derivative Test Without Using Graphs

Step 1: Obtain the formula for the derivative function.
Step 2: Use the formula for the derivative function to identify the critical numbers for the function.
Step 3: Pick a "test number" between each critical number and determine the output sign of the derivative function at each test number.
Step 4: Apply the First Derivative Test to classify the critical numbers.

Example 1. Without graphing, identify the input values where the local minimum and local maximum outputs occur for the function below.

$$y = f(x) = (x^2 - 9)^{2/3}$$

Solution. First, construct the formula for the derivative function. Observe that the Chain Rule tells us

$$f'(x) = \frac{4x}{3\sqrt[3]{x^2 - 9}}$$

The output of the derivative function will be 0 when $x = 0$; and the output of the derivative function will be undefined when $x = \pm 3$. Therefore, the function f has three critical numbers.

Now, these three critical numbers divide the real number line into four sets, namely the sets

$$-\infty < x < 3 \qquad -3 < x < 0 \qquad 0 < x < 3 \qquad 3 < x < +\infty$$

The derivative function must have the same output sign for all numbers in each set; the output sign can only change at the endpoints separating the sets. We can determine the output sign of the derivative function on each set by considering a single "test number" from each set.

- From the set $-\infty < x < 3$, consider the "test number" $x = -4$. (Any number from this set will do.) Observe that

$$f'(-4) = \frac{4(-4)}{3\sqrt[3]{(-4)^2 - 9}} = -\frac{16}{3\sqrt[3]{7}} < 0$$

- From the set $-3 < x < 0$, consider the "test number" $x = -1$. (Any number from this set will do.) Observe that

$$f'(-1) = \frac{4(-1)}{3\sqrt[3]{(-1)^2 - 9}} = \frac{4}{3\sqrt[3]{10}} > 0$$

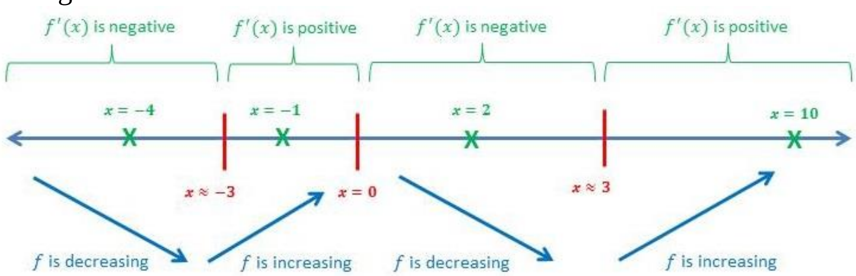
- From the set $0 < x < 3$, consider the "test number" $x = 2$. (Any number from this set will do.) Observe that

$$f'(2) = \frac{4(2)}{3\sqrt[3]{(2)^2 - 9}} = -\frac{8}{3\sqrt[3]{5}} < 0$$

- From the set $3 < x < +\infty$, consider the "test number" $x = 10$. (Any number from this set will do.) Observe that

$$f'(10) = \frac{4(10)}{3\sqrt[3]{(10)^2 - 9}} = \frac{40}{3\sqrt[3]{91}} > 0$$

We now have information about the behavior of the derivative function (and hence the behavior of the function f) as we move across each critical number from left to right. This information can best be visualized using a diagram.



1. As we move across the critical number $x = -3$, the output sign of the derivative function changes from negative to positive. The First Derivative Test tells us that the function f has a local minimum output at $x = -3$.

2. As we move across the critical number $x = 0$, the output sign of the derivative function changes from positive to negative. The First Derivative Test tells us that the function f has a local maximum output at $x = 0$.

3. As we move across the critical number $x = 3$, the output sign of the derivative function changes from negative to positive. The First Derivative Test tells us that the function f has a local minimum output at $x = 3$.

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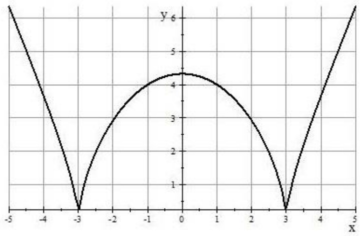
The graph of the function f from Example 1 confirms the analysis. Note the kinks in the graph at the input values $x = \pm 3$.



Problem 1. Without graphing the function, identify the input values where the local minimum and local maximum outputs occur for the function below.

$$y = f(x) = (x^2 - 4x)^{3/5}$$

Problem 2. Without graphing the function, identify the input values where the local minimum and local maximum outputs occur for the function below.

$$y = f(x) = (2x - 1)^3$$

Second Derivative

Let $y = f(x)$ be a function. The *second derivative* function for the function f is defined to be the derivative function for the derivative function of f. The second derivative function provides the instantaneous rate of change for the derivative function at any input value. The second derivative function is commonly denoted by

$$a = f''(x) \qquad \text{OR} \qquad a = \frac{d}{dx}\left[\frac{df}{dx}\right] \qquad \text{OR} \qquad a = \frac{d^2 f}{dx^2}$$

Example 1. Construct the second derivative function for $y = f(x) = x\cos(x)$.

Solution. The second derivative function is the derivative of the derivative function for f. Observe that the product rule tells us

$$f'(x) = \cos(x) - x\sin(x)$$

$$f''(x) = \frac{d}{dx}[\cos(x) - x\sin(x)] = -\sin(x) - \big(\sin(x) + x\cos(x)\big) = -2\sin(x) - x\cos(x)$$

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Problem 3. Construct a formula for the second derivative function of the function

$$y = f(x) = \cos(x) + \ln(x)$$

Problem 4. Construct a formula for the second derivative function of the function

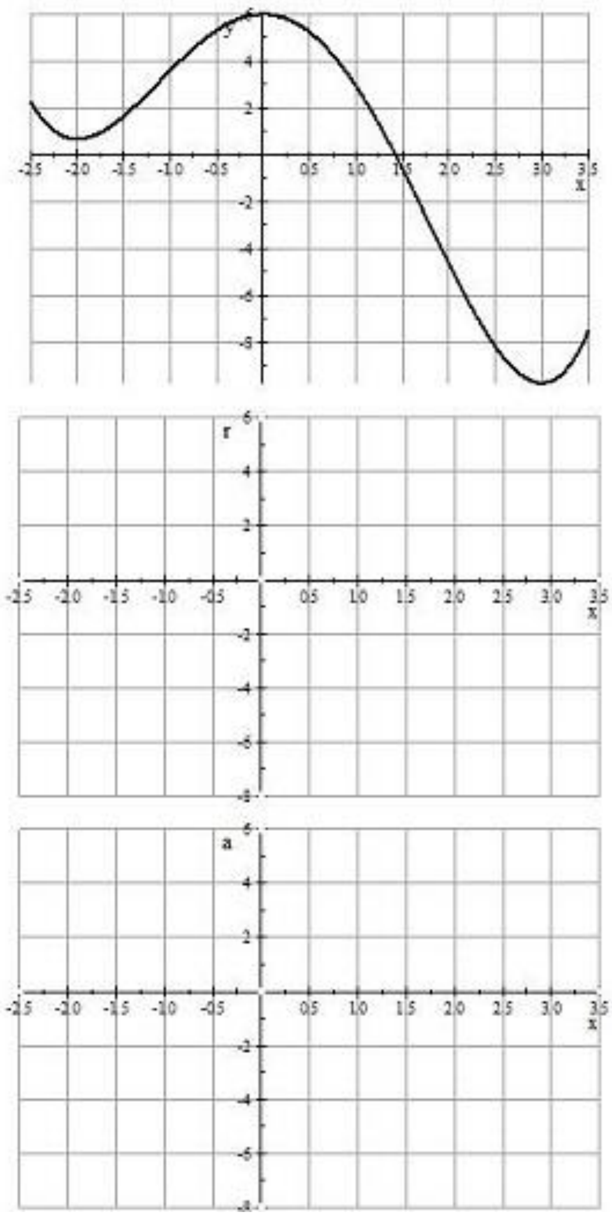
$$y = f(x) = e^x \sin(x)$$

Problem 5. Construct a formula for the *third* derivative function of the function

$$y = f(x) = \cos(x) + \ln(x)$$

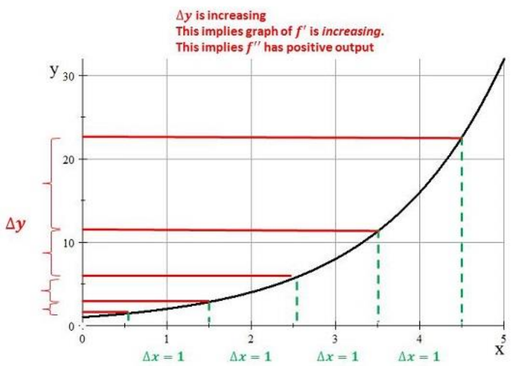
The third derivative function for a function $y = f(x)$ is often denoted by $j = f'''(x)$ or by $j = f^{(3)}(x)$.

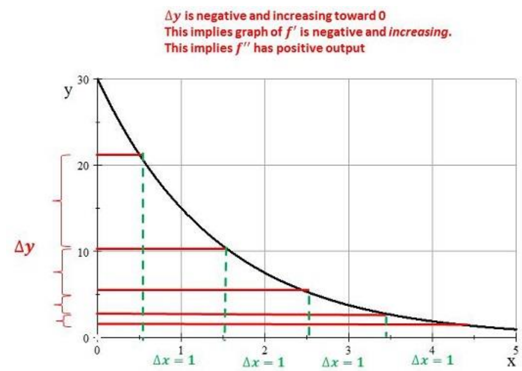
Problem 6. The diagram below shows the graph of a function $y = f(x)$. On the first grid, sketch the graph of the derivative function $r = f'(x)$ for the function f. Use your sketch of the derivative function f' to sketch the second derivative function $a = f''(x)$.



Problem 7. There is a relationship between the graph of the second derivative function and the input intervals where the graph of f is concave up or concave down. Describe what this relationship is.

The output of the second derivative function for a function $y = f(x)$ tells us the instantaneous rate of change *of the rate of change* for the function f. Because of this, the second derivative gives us information about the *concavity* of the function f.





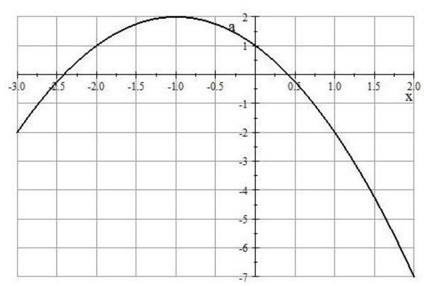
- If a function $y = f(x)$ is increasing at an increasing rate, then the second derivative function will have positive output, and the graph of the function f will be concave up.
- If a function $y = f(x)$ is decreasing at an increasing rate, then the second derivative function will have positive output, and the graph of the function f will be concave up.

- If a function $y = f(x)$ is increasing at a decreasing rate, then the second derivative function will have negative output, and the graph of f will be concave down.
- If a function $y = f(x)$ is decreasing at a decreasing rate, then the second derivative function will have negative output, and the graph of f will be concave down.

The output sign of the second derivative function on an input interval signals the concavity of the function on that input interval.

- If the output of the second derivative function is positive on an input interval, then the function is concave up on that input interval.
- If the output of the second derivative function is negative on an input interval, then the function is concave down on that input interval.

Problem 8. The diagram below shows the graph of the second derivative function for a function $y = f(x)$. On which input intervals will the graph of the function f be concave up? On which input intervals will the graph of the function f be concave down? At what input values will the function f have an inflection point?



Graph of Second Derivative Function

Second Derivative Test

Suppose the second derivative function for $y = f(x)$ is defined at some input value $x = a$. (We say that f is *twice differentiable* at $x = a$ in this case.)

- If the tangent line to the graph of f is horizontal at $(a, f(a))$ and $f''(a) < 0$, then the function f has a local maximum output at $x = a$.
- If the tangent line to the graph of f is horizontal at $(a, f(a))$ and $f''(a) > 0$, then the function f has a local minimum output at $x = a$.

The Second Derivative Test provides another way to identify local extreme outputs for a function f. The Second Derivative Test is not as powerful as the First Derivative Test, since it cannot be applied to all critical numbers for a function f --- it cannot be applied to critical numbers where the derivative function f' is undefined. The Second Derivative Test is only practical when the formula for the second derivative function is relatively easy to compute.

Example 2. Use the Second Derivative Test to classify the local extreme outputs for the function

$$y = f(x) = x^3 - 12x + 5$$

Solution. The Second Derivative Test requires that we first identify the critical numbers for the function f. Observe that

$$r = f'(x) = 3x^2 - 12$$

There are no input values where the derivative function will be undefined. Observe that

$$f'(x) = 0 \quad \Rightarrow \quad 3x^2 - 12 = 0 \quad \Rightarrow x = \pm 2$$

Thus, the function f has two critical numbers; and we know that the tangent line to the graph of f is horizontal at both. Now, observe that

$$a = f''(x) = 6x$$

Since $f''(-2) < 0$ and $f''(2) > 0$, the Second Derivative Test tells us that the function f will have a local maximum output at the input value $x = -2$ and will have a local minimum output at the input value $x = 2$...

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Problem 9. Use the Second Derivative Test to classify the local extreme outputs for the function

$$y = f(x) = x^4 - 4x^3 - 1$$

HOMEWORK: Section 4.3 (Pages 301-302) Problems 9, 10, 11, 17, 19, 20, 25, 27, 29, 35, 38, 65, 66, 67