BIG SHEET OF DERIVATIVE PRACTICE

Here are some basic facts to remember.

- A function $y = f(x)$ is *differentiable* at an input value $x = a$ provided

  1. The function is defined at $x = a$; that is $(a, f(a))$ is a point on the graph of f.
  2. When you zoom in repeatedly on the graph of f, focused on the point $(a, f(a))$, the graph of f looks more and more like a straight line.

  This straight line is called the *tangent line* to the graph of f at the point $(a, f(a))$.

- The slope of the tangent line to the graph of a function $y = f(x)$ at the point $(a, f(a))$ is called the *derivative* of the function f at $x = a$. The derivative of f at $x = a$ is denoted by the symbol $f'(a)$. The process of determining $f'(a)$ is called *differentiation.*

- The formula for the tangent line to the graph of $y = f(x)$ at the point $(a, f(a))$ is always given by $y - f(a) = f'(a)\,[x - a]$.

- The derivative of a function f at the at the input value $x = a$ can always be found by evaluating the limiting process

$$\lim_{x \longrightarrow a} \frac{f(x) - f(a)}{x - a} \qquad \text{OR} \qquad \lim_{h \longrightarrow 0} \frac{f(a+h) - f(a)}{h}$$

- The *derivative function* for a function $y = f(x)$ is a new function whose output is slope of the tangent line to f for any input value x. The derivative function for f is denoted by

$$f' \qquad \text{OR} \qquad \frac{df}{dx}$$

- If a function f has a relative maximum or minimum output at $x = a$, then $f'(a)$ will either be undefined or will be equal to 0.

- If a function f has an inflection point at $x = b$, then $f'(a)$ will be a relative maximum or minimum output for the derivative function f'.

- If the graph of f is increasing on an interval, then the graph of f' will be above the horizontal axis on this interval.

- If the graph of f is decreasing on an interval, then the graph of f' will be below the horizontal axis on this interval.

Using the limit definition to differentiate a function f is a very tedious process. Fortunately, this limiting process displays consistent properties that allow us to take shortcuts. These shortcuts fall into two classes.

SPECIAL RULES: These shortcuts tell us how to differentiate functions that are given by specific formulas.

GENERAL RULES: These shortcuts tell us how to differentiate combinations of functions.

When confronted with a function f we want to differentiate, we always follow the same stratetgy.

1. Use the General Rules to break up the derivative process for f into simpler and simpler derivative processes until we are faced only with derivatives of specific formulas.
2. Apply the Special Rules to differentiate the specific formulas.

SPECIFIC DERIVATIVE FORMULAS (Sections 3.1 - 3.3)

1. The derivative function for any constant function is the zero-function. That is, $\frac{d}{dx}[K] = 0$ for any fixed real number K.
2. If n is any nonzero integer, then $\frac{d}{dx}[x^n] = nx^{n-1}$.
3. If a is any positive real number, then $\frac{d}{dx}[a^x] = a^x \cdot \ln(a)$.
4. We have $\frac{d}{dx}[\sin(x)] = \cos(x)$, $\frac{d}{dx}[\cos(x)] = -\sin(x)$, and $\frac{d}{dx}[\tan(x)] = \sec^2(x)$.

GENERAL RULES (Sections 3.1 - 3.3)

Constant Multiple Rule: If $y = f\ (x)$ is a differentiable function and K is any fixed real number, then

$$\underbrace{(K \cdot f)' = K \cdot f'}_{\text{Prime Notation}} \qquad \text{OR} \qquad \underbrace{\frac{d}{dx}[K \cdot f(x)] = K \cdot \frac{d}{dx}[f(x)]}_{\text{Differential Notation}}$$

Sum Rule: If $y = f(x)$ and $y = g(x)$ are any differentiable functions, then the derivative function for $f + g$ is the sum of the derivative functions for f and g. In symbols,

$$(f+g)' = f' + g' \qquad \text{OR} \qquad \frac{d}{dx}[f(x) + g(x)] = \frac{d}{dx}[f(x)] + \frac{d}{dx}[g(x)]$$

Product Rule: If $y = g(x)$ and $y = h(x)$ are differentiable functions, then their product $f = g \cdot h$ is also differentiable, and

$$\underbrace{(g \cdot h)' = g \cdot h' + h \cdot g'}_{\text{Prime Notation}} \qquad \text{OR} \qquad \underbrace{\frac{d}{dx}[g(x) \cdot h(x)] = g(x) \cdot \frac{d}{dx}[h(x)] + h(x) \cdot \frac{d}{dx}[g(x)]}_{\text{Differential Notation}}$$

Quotient Rule: If $y = g(x)$ and $y = h(x)$ are differentiable functions, then their quotient $f = \frac{g}{h}$ is also differentiable as long as f is defined, and

$$\underbrace{\left(\frac{g}{h}\right)' = \frac{h \cdot g' - g \cdot h'}{h^2}}_{\text{Prime Notation}} \qquad \text{OR} \qquad \underbrace{\frac{d}{dx}\left[\frac{g(x)}{h(x)}\right] = \left(\frac{1}{[h(x)]^2}\right)\left(h(x) \cdot \frac{d}{dx}[g(x)] - g(x) \cdot \frac{d}{dx}[h(x)]\right)}_{\text{Differential Notation}}$$

Example 1 *Differentiate the function* $g(a) = 2a^{-3} + 5^a \sin(a) + 9$.

Solution. The formula for g is constructed from sums and products of functions whose derivatives we know. Therefore, we will use the General Rules to break up the derivative process for g into simpler and simpler derivative processes, until we are only left with the derivatives of specific formulas.

$$\begin{aligned}
g'(a) &= \frac{d}{da}\left[2a^{-3} + 5^a \sin(a) + 9\right] \\
&= \frac{d}{da}\left[2a^{-3}\right] + \frac{d}{da}\left[5^a \sin(a)\right] + \frac{d}{da}[9] && \text{Apply Sum Rule} \\
&= 2\frac{d}{da}\left[a^{-3}\right] + \frac{d}{da}\left[5^a \sin(a)\right] + \frac{d}{da}[9] && \text{Apply Constant Multiple Rule} \\
&= 2\frac{d}{da}\left[a^{-3}\right] + \left(5^a \cdot \frac{d}{da}[\sin(a)] + \sin(a) \cdot \frac{d}{da}[5^a]\right) + \frac{d}{da}[9] && \text{Apply Product Rule} \\
&= 2 \cdot \left(-3a^{-4}\right) + 5^a \cdot \cos(a) + \sin(a) \cdot 5^a \ln(5) + 0 && \text{Apply Specific Formulas} \\
&= -\frac{6}{a^4} + 5^a\left[\cos(a) + \ln(5)\sin(a)\right]
\end{aligned}$$

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EXTRA PROBLEMS FOR SECTIONS 3.1 - 3.3

Differentiate the following functions using the General Rules and the Specific Formulas.

(1) $f(x) = 3$ (2) $g(y) = 4y^2 - 1$ (3) $h(c) = 3c\sin(c)$

(4) $f(a) = 5a^{-4} + 2\cos(a)$ (5) $g(z) = \sin(z)\cos(z)$ (6) $h(u) = 4e^u - u \cdot 3^u$

(7) $f(t) = t\cos(t) + 3$ (8) $g(w) = w^2 \tan(w)$ (9) $h(p) = \pi - 2p\tan(p)$

(10) $f(r) = e - 3r^{-1} + 4r^3 \cdot 5^r$ (11) $g(s) = \left(4s^2 - 1\right)^2$ (12) $h(v) = \left(v^3 + 5v\right)^2$

(13) $f(b) = \left(2b^2 - b + 1\right)^{-1}$ (14) $g(k) = \dfrac{k^2}{k-1}$ (15) $h(n) = \dfrac{\sin(n)}{3n+5}$

16. What is the formula for the tangent line to $h(u) = u\cos(u)$ at the point $(0, h(0))$?

17. For what values of y will the tangent line to the graph of $g(y) = \dfrac{y^2}{1+y}$ have slope -1?

18. For what values of x will the tangent line to the graph of $f(x) = (4x-8)^2$ be horizontal?

ANSWERS:

1. Apply Constant Formula — $f'(x) = 0$
2. Apply Sum Rule followed by specific formulas — $g'(y) = 8y$
3. Apply Constant Multiple Rule followed by Product Rule followed by specific formulas — $h'(c) = 3\sin(c) + 3c\cos(c)$

4. Apply Sum Rule followed by Constant Multiple Rule followed by specific formulas — $f'(a) = -20a^{-5} - 2\sin(a)$

5. Apply Product Rule followed by specific formulas — $g'(z) = \cos^2(z) - \sin^2(z)$

6. Apply Sum Rule followed by Constant Multiple Rule followed by Product Rule followed by specific formulas — $h'(u) = 4e^u - 3^u(1 + u\ln(3))$

7. Apply Sum Rule followed by Product Rule followed by specific formulas — $f'(t) = \cos(t) - t\sin(t)$

8. Apply Product Rule followed by specific formulas — $g'(w) = 2w\tan(w) + w^2\sec^2(w)$

9. Apply Sum Rule followed by Constant Multiple Rule followed by Product Rule followed by specific formulas — $h'(p) = -2\tan(p) - 2p\sec^2(p)$

10. Apply Sum Rule followed by Constant Multiple Rule followed by Product Rule followed by specific formulas — $f'(r) = 3r^{-2} + 5^r\left(12r^2 + 4r^3\ln(5)\right)$

11. Apply Product Rule followed by Sum Rule followed by Constant Multiple Rule followed by specific formulas — $g'(s) = 16s(4s^2 - 1)$

12. Apply Product Rule followed by Sum Rule followed by Constant Multiple Rule followed by specific formulas — $h'(v) = (6v^2 + 10)(v^3 + 5v)$

13. Apply Quotient Rule followed by Sum Rule followed by Constant Multiple Rule followed by specific formulas — $f'(b) = -\dfrac{4b-1}{\left(2b^2 - b + 1\right)^2}$

14. Apply Quotient Rule followed by Sum Rule followed by specific formulas — $g'(k) = \dfrac{k(k-2)}{(k-1)^2}$

15. Apply Quotient Rule followed by Sum Rule followed by Constant Multiple Rule followed by specific formulas — $h'(n) = \dfrac{(3n+5)\cos(n) - 3\sin(n)}{(3n+5)^2}$

16. The point of tangency will be $(0,0)$. Now, $h'(u) = \cos(u) - u\cos(u)$. Therefore, the slope of the tangent line will be $h'(0) = 1$. The equation of the tangent line will be $y = u$.

17. First, note that

$$g'(y) = \frac{y^2 + 2y}{(1+y)^2}$$

Now, we want $g'(y) = -1$. Observe

$$\begin{aligned} -1 = \frac{y^2+2y}{(1+y)^2} \quad &\Longrightarrow \quad -(1 + 2y + y^2) = y^2 + 2y \\ &\Longrightarrow \quad 0 = 1 + 4y + 2y^2 \\ &\Longrightarrow \quad y = -1 \pm \frac{\sqrt{2}}{2} \qquad \text{(Apply Quadratic Formula)} \end{aligned}$$

18. First note that $f'(x) = 32x - 64$. We want $f'(x) = 0$. Observe

$$32x - 64 = 0 \Longrightarrow x = 2$$

THE CHAIN RULE

The last of the general rules for differentiation is the *Chain Rule*. The Chain Rule is a shortcut to the limit definition of the differentiation process that tells us how to find the derivative of composite functions.

CHAIN RULE: If f is a differentiable function of u and u is a differentiable function of x, then

$$\frac{df}{dx} = \frac{df}{du} \cdot \frac{du}{dx}$$

Example 2 *Differentiate the function* $g(x) = \cos^2(x^3)$.

Solution. The function g is actually a composition of three functions, since

$$f(x) = \left[\cos(x^3)\right]^2$$

We assign a function name to each formula in the composition. Working from the inside out, let

- $v(x) = x^3$ so that $\dfrac{dv}{dx} = 3x^2$
- $u(w) = \cos(v)$ so that $\dfrac{du}{dv} = -\sin(v)$
- $f(u) = v^2$ so that $\dfrac{df}{du} = 2u$

Observe that $g(x) = f(u(v(x))$. Now, according to the Chain Rule, we have

$$\begin{aligned} \frac{d}{dx}\left[\cos^2(x^3)\right] &= \frac{df}{du} \cdot \frac{du}{dv} \cdot \frac{dv}{dx} \\ &= [2u] \cdot [-\sin(v)] \cdot \left[3x^2\right] \\ &= [2\cos(v)] \cdot [-\sin(v)] \cdot \left[3x^2\right] \\ &= \left[2\cos(x^3)\right] \cdot \left[-\sin(x^3)\right] \cdot \left[3x^2\right] \\ &= -6x^2\cos(x^3)\sin(x^3) \end{aligned}$$

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Example 3 *Differentiate the formula* $\cos(y) + \sin(x) + 8 = z$ *with respect to the variable* t.

Solution. Observe that

$$\begin{aligned} \frac{dz}{dt} &= \frac{d}{dt}\left[\cos(y) + \sin(x) + 8\right] \\ &= \frac{d}{dt}[\cos(y)] + \frac{d}{dt}[\sin(x)] + \frac{d}{dt}[8] && \text{Apply Sum Rule} \\ &= -\sin(y)\frac{dy}{dt} + \cos(x)\frac{dx}{dt} + 0 && \text{Apply Chain Rule} \\ &= -\sin(y)\frac{dy}{dt} + \cos(x)\frac{dx}{dt} \end{aligned}$$

Since we do not have formulas for y and x in terms of the variable t, we cannot take this derivation any further.

The Chain Rule allows us to develop several more specific derivative formulas.

SPECIFIC DERIVATIVE FORMULAS (Sections 3.4 - 3.6)

- For any nonzero rational number r, we have $\dfrac{d}{dx}[x^r] = rx^{r-1}$.

- We have $\dfrac{d}{dx}[\ln(x)] = \dfrac{1}{x}$.

- As long as $-1 \leq x \leq 1$, we have $\dfrac{d}{dx}[\arcsin(x)] = \dfrac{1}{\sqrt{1-x^2}}$, where $y = \arcsin(x)$ is the principle inverse sine function.

- We have $\dfrac{d}{dx}[\arctan(x)] = \dfrac{1}{1+x^2}$, where $y = \arctan(x)$ is the principle inverse tangent function.

EXTRA PROBLEMS FOR SECTIONS 3.4 - 3.6

Use the Chain Rule to differentiate the following functions.

(1) $f(x) = \sin(4x+1)$	(2) $g(x) = \tan^2(x)$	(3) $h(x) = \left(x^2-1\right)^{-4}$
(4) $f(x) = \cos^4(x)$	(5) $g(x) = (5x-4)^{3/4}$	(6) $h(x) = \ln(\sin(x) - \cos(x))$
(7) $f(x) = \tan(\ln(x))$	(8) $g(x) = \arctan(x^3)$	(9) $h(x) = \arcsin(\arctan(x))$

10. Differentiate the equation $y = x^2 + \sin(x)$ with respect to the variable t.

11. Differentiate the equation $x^2 + y^2 = z^2$ with respect to the variable w.

12. Find a formula for $\dfrac{dx}{dt}$ if $t = x^2 + \tan(x)$.

13. Find a formula for $\dfrac{dy}{dx}$ if $y^3 - y = x^3$.

14. Find a formula for $\dfrac{dv}{du}$ if $v\sin(u) = u\sin(v)$

15. Differentiate the function $f(x) = \tan^3(x^{4/5})$.

16. Differentiate the function $f(x) = (\arcsin(8x-2))^4$.

17. Differentiate the function $f(x) = 3\sqrt{x} + 4x^2\ln(\cos(x))$. (Remember, $\sqrt{x} = x^{1/2}$.)

ANSWERS

1. Let $u(x) = 4x + 1$. We have $f'(x) = \frac{d}{du}[\sin(u)] \cdot \frac{d}{dx}[4x+1] = 4\cos(4x+1)$.

2. Let $u(x) = \tan(x)$. We have $g'(x) = \frac{d}{du}\left[u^2\right] \cdot \frac{d}{dx}[\tan(x)] = 2\tan(x)\sec^2(x)$.

3. Let $u(x) = x^2 - 1$. We have $h'(x) = \frac{d}{du}\left[u^{-4}\right] \cdot \frac{d}{dx}\left[x^2 - 1\right] = -\frac{8}{(x^2-1)^5}$.

4. Let $u(x) = \cos(x)$. We have $f'(x) = \frac{d}{du}\left[u^4\right] \cdot \frac{d}{dx}[\cos(x)] = -4\cos^3(x)\sin(x)$.

5. Let $u(x) = 5x - 4$. We have $g'(x) = \frac{d}{du}\left[u^{3/4}\right] \cdot \frac{d}{dx}[5x-4] = \frac{15}{4}(5x-4)^{-1/4}$.

6. Let $u(x) = \sin(x) - \cos(x)$. We have $h'(x) = \frac{d}{du}[\ln(u)] \cdot \frac{d}{dx}[\sin(x) - \cos(x)] = \frac{\cos(x)+\sin(x)}{\sin(x)-\cos(x)}$.

7. Let $u(x) = \ln(x)$. We have $f'(x) = \frac{d}{du}[\tan(u)] \cdot \frac{d}{dx}[\ln(x)] = \frac{\sec^2(\ln(x))}{x}$.

8. Let $u(x) = x^3$. We have $g'(x) = \frac{d}{du}[\arctan(u)] \cdot \frac{d}{dx}\left[x^3\right] = \frac{3x^2}{1+x^6}$.

9. Let $u(x) = \arctan(x)$. We have $h'(x) = \frac{d}{du}[\arcsin(u)] \cdot \frac{d}{dx}[\arctan(x)] = \left[\left(1+x^2\right)\sqrt{1-\arctan^2(x)}\right]^{-1}$.

10. We have $\frac{dy}{dt} = (2x + \cos(x)) \cdot \frac{dx}{dt}$.

11. We have $x \cdot \frac{dx}{dw} + y \cdot \frac{dy}{dw} = z \cdot \frac{dz}{dw}$.

12. We have $\frac{dx}{dt} = \frac{1}{2x+\sec^2(x)}$.

13. We have $\frac{dy}{dx} = \frac{3x^2}{3y^2-1}$.

14. We have $\frac{dv}{du} = \frac{v\cos(u)-\sin(v)}{u\cos(v)-\sin(u)}$.

15. Let $w(x) = x^{4/5}$, let $v(w) = \tan(w)$, and let $u(v) = v^3$. We have
$$\frac{df}{dx} = \frac{du}{dv} \cdot \frac{dv}{dw} \cdot \frac{dw}{dx} = \frac{12}{5x^{1/5}} \tan^2\left(x^{4/5}\right) \sec^2\left(x^{4/5}\right)$$

16. Let $w(x) = 8x - 2$, let $v(w) = \arcsin(w)$, and let $u(v) = v^4$. We have
$$\frac{df}{dx} = \frac{du}{dv} \cdot \frac{dv}{dw} \cdot \frac{dw}{dx} = \frac{32\left[\arcsin(8x-2)\right]^3}{\sqrt{1-(8x-2)^2}}$$

17. We have $f'(x) = \frac{3}{2\sqrt{x}} + 8x\ln(\cos(x)) - 4x^2\tan(x)$.