ALGEBRA UNIT 4 — The Distributive Law

When we solved equations in the previous units, we almost always had to "combining like terms" as part of the process. Combining like terms is an example of a bigger principal in algebra called the *distributive law*:

- **The Distributive Law:** If A, B and C are expressions, then $(B+C)A = BA+CA$. This rule *does not work* if $B+C$ is raised to any power other than 1.

We say that the expression A *distributes over the sum* $B+C$. Notice that BA and CA are really like terms with coefficients B and C; consequently, the Distributive Law is just a way of saying that we combine like terms by adding the coefficients. Consequently, we have been using the Distributive Law all along.

However, the Distributive Law works both ways. We have been using it to combine like terms; we can also use it to expand a product into a sum of like terms (with respect to the factor A). Expanding a product into a sum of like terms is very useful in solving many types of equations, as we will see.

Example 1 *Use the Distributive Law to expand the expression* $(x+4ty)\,y^2$ *into a sum of like terms.*

Solution. In this case, we let $A = y^2$, $B = x$, and $C = 4ty$. The like terms will all be with respect to $A = y^2$. Observe

$$\begin{aligned}
(x+4ty)\,y^2 &= (B+C)A \\
&= BA+CA && \text{Apply the Distributive Law} \\
&= xy^2 + (4ty)\,y^2 && \text{Sum of like terms with respect to } y^2 \\
&= xy^2 + 4ty^3
\end{aligned}$$

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Example 2 *Use the Distributive Law to expand the expression* $3\sqrt{x}\left(2+t^{-3}\right)$ *into a sum of like terms. Use only positive exponents in your final answer.*

Solution. In this case, $A = 3\sqrt{x}$, $B = 2$, and $C = t^{-3}$. It does not matter whether the expression A appears on the left or the right in the product; the Distributive Law tells us

$$\begin{aligned}
3\sqrt{x}\left(2+t^{-3}\right) &= A(B+C) \\
&= AB+AC && \text{Apply the Distributive Law} \\
&= \left(3\sqrt{x}\right)\cdot 2 + \left(3\sqrt{x}\right)\cdot t^{-3} && \text{Sum of like terms with respect to } 3\sqrt{x} \\
&= 6\sqrt{x} + \frac{3\sqrt{x}}{t^3}
\end{aligned}$$

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Example 3 *Use the Distributive Law to show that* $(2+y)^2 = 4+4y+y^2$.

Solution. First, observe that $(2+y)^2 = (2+y)\cdot(2+y)$. Now, let $A = 2+y$ be the left sum. (You could also let A be the right sum.) Break up the right sum by letting $B = 2$ and $C = y$. We then see that

$$\begin{aligned}
(2+y)^2 &= (2+y)\cdot(2+y) \\
&= A(B+C) \\
&= AB+AC && \text{Applying the Distributive Law} \\
&= (2+y)\cdot 2 + (2+y)\cdot y && \text{Sum of like terms with respect to } 2+y \\
&= (B+C)B + (B+C)C \\
&= BB+CB+BC+CC && \text{Applying the Distributive Law twice} \\
&= (2)(2)+(y)(2)+(2)(y)+yy \\
&= 4+4y+y^2
\end{aligned}$$

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There is an etension of the Distributive Rule that we could have applied in the previous problem.

General Distributive Rule If U and V are sums raised to the first power, then we expand the product UV by multiplying every term of U by every term of V and adding up all these products.

This rule makes expanding the product $(2+y)^2$ easier. Since there are two terms in the sum $2+y$, there will be four terms in the product $(2+y)(2+y)$, namely

$$2 \cdot 2 \qquad 2 \cdot y \qquad y \cdot 2 \qquad y \cdot y$$

We then expand $(2+y)^2$ by adding all up all of these products.

$$(2+y)^2 = 2 \cdot 2 + 2 \cdot y + y \cdot 2 + y \cdot y = 4 + 4y + y^2$$

Example 4 *Expand the product* $(3+y)\left(4 - \frac{1}{x}\right)$.

Solution. We start by multiplying every term in $3+y$ by every term in $4 - \frac{1}{x}$ to obtain four products

$$3 \cdot 4 \qquad 3 \cdot \left(-\frac{1}{x}\right) \qquad y \cdot 4 \qquad y \cdot \left(-\frac{1}{x}\right)$$

We complete the problem by adding up all these products.

$$\begin{aligned} (3+y)\left(4 - \frac{1}{x}\right) &= 3 \cdot 4 + 3 \cdot \left(-\frac{1}{x}\right) + y \cdot 4 + y \cdot \left(-\frac{1}{x}\right) \\ &= 12 - \frac{3}{x} + 4y - \frac{y}{x} \\ &= 12 + 4y - \frac{3+y}{x} \end{aligned}$$

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Example 5 *Expand the product* $(2 - 3y + z)(y - 2z)$.

Solution. Expanding this product will require six terms, namely

$$2 \cdot y \qquad 2 \cdot (-2z) \qquad (-3y) \cdot y \qquad (-3y) \cdot (-2z) \qquad z \cdot y \qquad z \cdot (-2z)$$

We complete the problem by adding up all of these products.

$$\begin{aligned} (2 - 3y + z)(y - 2z) &= 2 \cdot y + 2 \cdot (-2z) + (-3y) \cdot y + (-3y) \cdot (-2z) + z \cdot y + z \cdot (-2z) \\ &= 2y - 4z - 3y^2 + 6yz + yz - 2z^2 \\ &= 2y - 4z - 3y^2 + 7yz - 2z^2 \end{aligned}$$

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Exercises

1. Use $x = 1$ and $y = 2$ to show that $(x+y)^2$ is not always equal to $x^2 + y^2$. Find another set of values for x and y which make the equation $(x+y)^2 = x^2 + y^2$ false.

2. Use $x = 3$ and $y = 4$ to show that $\sqrt{x^2 + y^2}$ is not always equal to $x + y$. Find another set of values for x and y which make the equation $\sqrt{x^2 + y^2} = x + y$ false.

3. Use $x = -1$ and $y = 2$ to show that $3\left(x + y\right)^2$ is not always equal to $\left(3x + 3y\right)^2$. Find another set of values for x and y which make the equation $3\left(x + y\right)^2 = \left(3x + 3y\right)^2$ false.

Use the Distributive Rule or the General Distributive Rule to expand the following products. Use only positive exponents in your final answer.

(4) $4x^{-1}\left(3y + 7\right)$	(5) $\sqrt{xy}\left(2x + 3y^{-1}\right)$	(6) $3\left(1 - x\right)^2$
(7) $\left(1 - x\right)\left(1 + x\right)$	(8) $\left(3y + \sqrt{2}\right)\left(3y - \sqrt{2}\right)$	(9) $\left(\sqrt{w} + 2z\right)^2$
(10) $\left(c + d\right)\left(c - 2\right)^2$	(11) $\left(3 - 2y\right)^2\left(y - t\right)$	(12) $\left(3 - 2x\right)^3$

(13) By expanding both sides of the equation, show that $\left(\sqrt{3}x + \sqrt{3}y\right)^2 = 3\left(x + y\right)^2$.

(14) By expanding the left-hand side of the equation, show that $\left(A + B\right)\left(A + B\right) = A^2 + 2AB + B^2$ for any expressions A and B.

(15) By expanding the left-hand side of the equation, show that $\left(A + B\right)\left(A - B\right) = A^2 - B^2$ for any expressions A and B.

(16) Use the equation in Problem 15 to find expressions A and B such that $9 - 2x = (A + B)(A - B)$.

(17) By expanding the right-hand side of the equation, show that $A^3 - B^3 = (A - B)(A^2 + AB + B^2)$ for any expressions A and B.