Discrete versus Continuous Growth

A quantity is experiencing *discrete growth* (or decay) if there is always the same fixed time period between instances of growth. These fixed time periods are called *growth periods* (or decay periods), and the value of the quantity does not change during a growth period. Here are some examples of discrete exponential growth.

Example 1 *If you deposit \$1,000 dollars in an account that pays 3.2% annual interest compounded quarterly, then there will be four growth periods every year. These periods are all three months long, and during these periods, nothing happens to the money in your account.*

Example 2 *If the student enrollment at a high school is expected to decrease at an annual rate of 5% per year for the next ten years, then there is one decay period each year. Each period is twelve months long, and nothing happens to the enrollment during these periods.*

It seems reasonable to imagine that the amount money in your account does not change over a period of three months (assuming you don't draw anything out of the account). It is probably also reasonable to assume that the high school enrollment does not change (at least not much) after the first few days of a new school year. However, there are changing quantities for which discrete growth periods really *don't* make much sense.

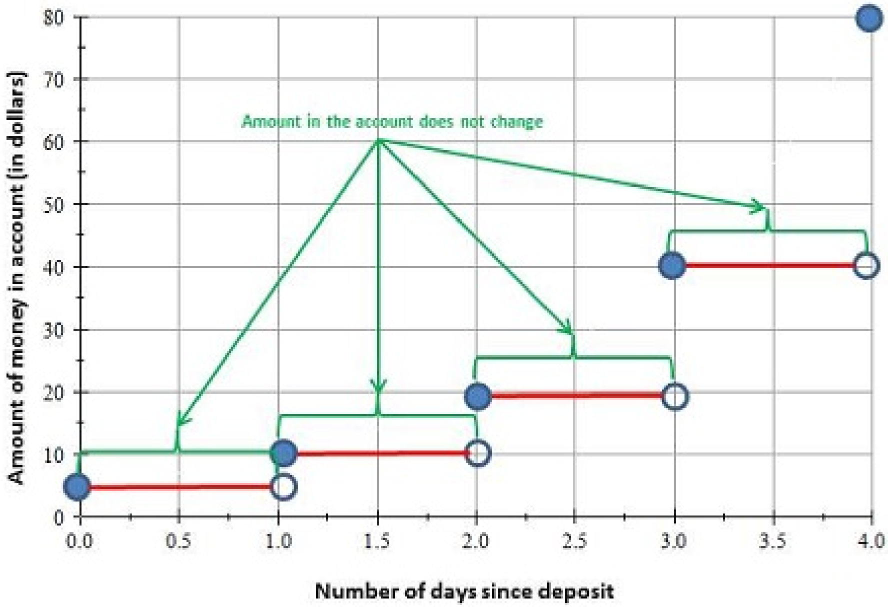
Example 3 *Consider a sample of bacteria. As time passes, individual bacteria cells will divide at random. There may be very brief time periods now and then when no cells are dividing, but there will never be a fixed time period where you can be certain the population is not changing.*

We say that the values of a quantity are changing *continuously* when there is never a fixed time period, no matter how short, where we can be certain the values of the quantity will remain constant. Graphs for discrete or continuous growth will look very different.

Example 4 *Suppose you deposit five dollars in an account that pays* 100% *interest at the end of each day. Let* t *be the number of days passed since the deposit, and let* A *be the amount of money in the account. If* f *is the function that gives* A *in terms of* t*, then what would the graph of* f *look like?*

Solution. The bank account is experiencing discrete growth — there is no change in the amount of money in the account until the end of each day. Therefore, when t is an *integer* (that is, when $t = 0, 1, 2, 3, ...$ etc.), we know that $f(t) = 5 \cdot 2^t$. However, when t is NOT an integer (for example, if $t = 1.5$), this output formula is not valid. The number $t = 1.5$ represents one and a half days after the deposit. The amount in the account is *constant* from the start of Day 1 to the end of Day 1 (it stays at the value $5 \cdot 2^1$ dollars).

Here is what the graph of f would look like. Note that $f(t) = 5 \cdot 2^t$ ONLY when t is an integer.



Accounts that earn compound interest are good examples of discrete exponential relationships. To get an understanding of how continuous growth is different from discrete growth, let's see what happens if we increase the annual compounding rate in a compound interest problem. This not only shortens the time periods when the amount in the account is constant, it also changes the growth factor in the output formula. The end result is a bit surprising.

1. Let's suppose you deposit one dollar in an account that pays 100% annual interest, and suppose you leave the money in this account for one year. If we let n be the number of compounding periods in that year, let P be the amount of money in the account after one year, and let f be the name of the function that gives the value of P in terms of the value of n, write down the formula that defines f. Use proper function notation.

2. Use your formula to fill in the following table. Keep at least five decimal place accuracy.

Value of n	1	4	12	52	365	525600	31536000
Value of P							

The amount of money in the account appears to be "settling down" to a fixed number as the compounding rate increases. As the compounding rate gets larger and larger, the length of the payment periods gets shorter and shorter. When we want to indicate that the values of a quantity grows larger and larger without bound, we do so by saying the values *approach infinity*. We can use the symbol

$$n \longrightarrow \infty$$

as a way to write "the values of n approach infinity." Now, as $n \longrightarrow \infty$, the output values of the function f settle down to a special number called *Euler's constant* which we represent by the special symbol e. This number is a nonterminating, nonrepeating decimal and has been computed to billions of decimal places. For our purposes, it is enough to say that $e \approx 2.7183$. We say that e is the *limit* of the values of $f(n)$ as $n \longrightarrow \infty$. We use the special symbol

$$\lim_{n \longrightarrow \infty} f(n) = e$$

to represent this.

3. Suppose you deposit \$3000 in an account that pays 4.7% annual interest, compounded every second (so there are 31,536,000 compounding periods per year).

(a) Let g be the function that gives the amount of money in the account as a function of the number t of years since the initial deposit. What is the growth factor for this function?

(b) Let b be the growth factor for the function g. Solve the equation $x^{.047} = b$ for the variable b. What do you notice?

4. Suppose you deposit \$12,500 into an account that pays 6.65% annual interest, compounded every tenth of a second (so there are 315,360,000 compounding periods per year).

(a) Let h be the function that gives the amount of money in the account as a function of the number t of years since the initial deposit. What is the growth factor for this function?

(b) Let b be the growth factor for the function h. Solve the equation $x^{.0665} = b$ for the variable b. What do you notice?

You should have noticed that in each problem, $x \approx e$. In other words, for the enormous number of compounding periods we considered per year, we have

$$g(t) \approx 3000 \left(e^{.047}\right)^t \qquad\qquad h(t) \approx 12500 \left(e^{.0665}\right)^t$$

With the very short growth periods resulting from the huge number of compounding periods per year, the amount of money in your account is almost experiencing *continuous* growth. This motivates the following definition.

Continuous Growth Formula

Suppose the values of a quantity Q experience continuous growth at a percentage rate r per unit of time (written as a decimal). If we let t represent the number of these units of time that have passed since the initial value of Q, then the function k that gives the values of Q as a function of the values of t is defined by

$$Q = k(t) = A\left(e^r\right)^t$$

where A represents the initial value of the quantity. (Continuous decay is handled by letting r be negative in the formula.)

5. Suppose you invest \$5000 into an account that pays 3.25% annual interest, compounded continuously. Write down the formula that gives the amount of money in the account as a function of the number of years since the initial deposit. Use proper function notation.

6. Suppose the mass of an eleven gram sample of bacteria experiences continuous decay at a rate of 11.36% per day. What is the mass of the sample after one week?