ALGEBRA UNIT 3 — How We Use Exponents

In Unit 2, we looked at ways to solve equations that are linear with respect to a variable. If the equation is not linear with respect to the variable we want to solve for, then we have to use different strategies from the one explored in Unit 2. We start the process of introducing some of these strategies by taking a look at the notion of *exponents*. Exponents were originally introduced as a shorthand for repeated multiplication, but their use has grown well beyond the shortand. However, the shorthand is the best place to begin an exploration of how we use exponents in modern algebra.

Fundamental Definition: Let A be any expression, and let n be a positive integer (natural number). The expression A^n is the product of A with itself n times. In other words,

$$A^n = \underbrace{A \cdot A \cdot A \cdot ... \cdot A}_{n \text{ times}}$$

We say that the expression A has been *raised to the nth power* and call n the *exponent* of the new expression A^n. For example

$$4^2 = 4 \cdot 4 \qquad x^4 = x \cdot x \cdot x \cdot x \qquad \left(\frac{2t-1}{t+1}\right)^3 = \left(\frac{2t-1}{t+1}\right) \cdot \left(\frac{2t-1}{t+1}\right) \cdot \left(\frac{2t-1}{t+1}\right)$$

Raising expressions to powers obey some common sense rules. For example, if A and B are any expressions and m and n are positive integers, then

$$\begin{aligned} A^m \cdot A^n &= \underbrace{A \cdot A \cdot A \cdot ... \cdot A}_{m \text{ times}} \cdot \underbrace{A \cdot A \cdot A \cdot ... \cdot A}_{n \text{ times}} \\ &= \underbrace{A \cdot A \cdot A \cdot ... \cdot A}_{m+n \text{ times}} \\ &= A^{m+n} \end{aligned}$$

$$\begin{aligned} (A^m)^n &= \left(\underbrace{A \cdot A \cdot A \cdot ... \cdot A}_{m \text{ times}}\right)^n \\ &= \underbrace{\underbrace{A \cdot A \cdot A \cdot ... \cdot A}_{m \text{ times}} \cdot \underbrace{A \cdot A \cdot A \cdot ... \cdot A}_{m \text{ times}} \cdot ... \cdot \underbrace{A \cdot A \cdot A \cdot ... \cdot A}_{m \text{ times}}}_{n \text{ times}} \\ &= \underbrace{A \cdot A \cdot A \cdot ... \cdot A}_{mn \text{ times}} \\ &= A^{mn} \end{aligned}$$

$$\begin{aligned} A^m \cdot B^m &= \left(\underbrace{A \cdot A \cdot A \cdot ... \cdot A}_{m \text{ times}}\right) \cdot \left(\underbrace{B \cdot B \cdot B \cdot ... \cdot B}_{m \text{ times}}\right) \\ &= \underbrace{(AB) \cdot (AB) \cdot (AB) \cdot ... \cdot (AB)}_{m \text{ times}} \\ &= (AB)^m \end{aligned}$$

This shorthand for repeated multiplication proved so useful that, over the centuries, we found ways to expand it to cover more sophisticated algebra operations. Taking roots is one example. We say that an expression A is the *nth root* of an expression B provided $B = A^n$ is a true statement. Since the Middle Ages, we have let $\sqrt[n]{B}$ represent the nth root of B when it exists. (The special symbol $\surd$ is actually a stylized "R" and stands for *radix* which is the Latin word for "root.")

Now, if B is an expression and $\sqrt[n]{B}$ exists, then based on the way we have defined the process of "raising to a power" we know

$$B = B^1 = \left(\sqrt[n]{B}\right)^n$$

Since $1 = \left(\frac{1}{n}\right) \cdot n$, it seems reasonable to extend the process of "raising to a power" to include the process of "taking roots" simply by letting

$$B^{1/n} = \sqrt[n]{B}$$

This definition brings the process of "taking roots" in line with the common-sense observation that $(A^m)^n = A^{mn}$ for positive integer exponents. For this reason, we make the following definition.

Definition of Roots as Powers: If B is any expression, n is a positive integer, and the nth roof of B exists, then we let $B^{1/n}$ represent this root. We read the expression $B^{1/n}$ as "B raised to the $1/n$ power."

Some common roots have special names that we usually don't change. For example, the 2nd root of an expression B (denoted by $B^{1/2}$ when it exists) is traditionally called the "square root" of B. It is common to write $\sqrt{B}$ instead of either $\sqrt[2]{B}$ or $B^{1/2}$.

The nth root of an expression B might not exist. For example, there is no real number A whose square is the real number $B = -1$. Consequently, the square root of -1 does not exist. However, the third root (or *cube* root as it is commonly called) for $B = -1$ does exist — it is simply $A = -1$. Since the third root of $B = -1$ exists, it would be correct to write $(-1)^{1/3} = -1$.

Example 1 *Interpret and evaluate the expression* $16^{3/2}$.

Solution. The exponent $3/2$ is a ratio of two integers (these are called *rational numbers*). We interpret this power using the rules for multiplying fractions. We know

$$\frac{3}{2} = \left(\frac{1}{2}\right) \cdot 3$$

With this in mind, it seems reasonable to interpret this power in the following way

$$16^{3/2} = 16^{\left(\frac{1}{2}\right) \cdot 3} = \left(16^{1/2}\right)^3$$

In words, $16^{3/2}$ represents the cube of the square root of 16. Now, since $16^{1/2} = 4$, we know that

$$16^{3/2} = \left(16^{1/2}\right)^3 = (4)^3 = 64$$

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As we have just seen, negative numbers cause us problems when we try to determine some roots but not others. We sometimes say that we are taking an *odd* root when we are looking for the nth root of an expression and n is an odd integer. Likewise, we say that we are taking an *even* root when n is even. We will always have a problem trying to take an even root of a negative number. The reason this will be a problem is because an even power of any expression must always be nonnegative.

Example 2 *For what values of t will the expression $\sqrt{2t+1}$ be undefined?*

Solution. We know that an expression can be undefined when variables appear in its denominator. If an expression has a variable under a square root (which is an even root), then it will be undefined whenever the variable expression under the square root is negative. (Square roots of negative numbers are undefined.)

Now, $\sqrt{2t+1}$ has the expression $2t + 1$ under a square root. For what values of t will this expression be negative? Observe

$$\begin{aligned}
2t + 1 < 0 \quad &\Longrightarrow \quad (2t+1) - 1 < 0 - 1 && \text{Subtract 1 from both sides} \\
&\Longrightarrow \quad 2t < -1 && \text{Combine like terms} \\
&\Longrightarrow \quad \frac{2t}{2} < -\frac{1}{2} && \text{Divide both sides by 2} \\
&\Longrightarrow \quad t < -\frac{1}{2} && \text{Apply Cancellation Law}
\end{aligned}$$

The expression under the square root will be negative when $t < -1/2$; hence, the expression $\sqrt{2t+1}$ will be undefined for this range of values for t. For example, if we let $t = -1$ (which is a value less than $-1/2$), then

$$\sqrt{2t+1} = \sqrt{2(-1)+1} = \sqrt{-1} \quad \text{(Undefined)}$$

We would have the same problem for *any* value of t that is strictly less than $-1/2$.

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Example 3 *For what values of x will $(3x+5)^{1/3}$ be undefined?*

Solution. We are dealing with an odd root in this expression. Odd roots are defined when the expression they contain is negative; therefore we do not have to worry about solving the inequality $3x + 5 < 0$. Since there is no variable in the denominator of this expression (there is no denominator at all), we may conclude that $(3x+5)^{1/3}$ is defined for all values of x.

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Example 4 *For what values of a will the expression $\dfrac{a+\sqrt[4]{8-2a}}{3a-6}$ be undefined?*

Solution. First, notice that there is a variable expression in the denominator; hence we must be concerned about division by 0. Observe

$$\begin{aligned}
3a-6=0 \quad &\Longrightarrow \quad (3a-6)+6=0+6 && \text{Add 6 to both sides}\\
&\Longrightarrow \quad 3a=6 && \text{Combine like terms}\\
&\Longrightarrow \quad \frac{3a}{3}=\frac{6}{3} && \text{Divide both sides by 3}\\
&\Longrightarrow \quad a=2 && \text{Apply Cancellation Law}
\end{aligned}$$

Consequently, we know the expression will be undefined when $a = 2$ (because this value of a makes the denominator equal 0). Now, there is also an even root in the numerator of the expression, namely $\sqrt[4]{8-2a}$. The expression under this root cannot be negative. Observe

$$\begin{aligned}
8-2a<0 \quad &\Longrightarrow \quad (8-2a)+2a<0+2a && \text{Add } 2a \text{ to both sides}\\
&\Longrightarrow \quad 8<2a && \text{Combine like terms}\\
&\Longrightarrow \quad \frac{8}{2}<\frac{2a}{2} && \text{Divide both sides by 2}\\
&\Longrightarrow \quad 4<a && \text{Apply Cancellation Law}
\end{aligned}$$

We now see that the expression will be undefined when $a = 2$ *or* whenever $a > 4$. We do have to worry about any other part of this expression.

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We can extend the process of "raising to a power" to include division as well. Let A be any expression that is not equal to 0 and observe that

$$\begin{aligned}
\frac{A^5}{A^3} &= \frac{A\cdot A\cdot A\cdot A\cdot A}{A\cdot A\cdot A}\\
&= \frac{\cancel{A}\cdot\cancel{A}\cdot\cancel{A}\cdot A\cdot A}{\cancel{A}\cdot\cancel{A}\cdot\cancel{A}}\\
&= A\cdot A\\
&= A^2
\end{aligned}$$

We had to require that A be nonzero because we never allow division by 0. Now, since $A^2 = A^{5-3}$, and since the rules for multiplying fractions tell us that

$$\frac{A^5}{A^3} = A^5 \cdot \left(\frac{1}{A^3}\right)$$

it seems reasonable to let $A^{-3} = \dfrac{1}{A^3}$.

Definition of Division as Powers If A is any expression that is not equal to 0 and n is a positive integer, then we let $A^{-n} = \dfrac{1}{A^n}$. Also, we let $A^{-1/n} = \dfrac{1}{A^{1/n}}$ as long as the nth root of A exists.

We have extended the simple shorthand for repeated multiplication to include the process of taking roots and the process of division. There is one last definition we should make to bring the concept of "raising to a power" as far as we can take it right now. As long as A is a nonzero expression, we know

$$1 = \frac{A}{A} = A \cdot \left(\frac{1}{A}\right) = A^1 \cdot A^{-1}$$

Now, since $0 = 1 - 1$, it seems reasonable to let $A^0 = 1$ (as long as A is not equal to 0).

We have now developed the so-called *Laws of Exponents* almost as far as we will need to take them.

THE LAWS OF EXPONENTS

Let A be any expression and let n be a positive integer.

1. We let $A^0 = 1$ as long as A is not equal to 0.
2. We let A^n be the product of A with itself n times.
3. We let $A^{1/n}$ be the nth root of A when this root exists.
4. We let $A^{-n} = \dfrac{1}{A^n}$ as long as A is not equal to 0.

   Let A and B be any expressions and let u and v be any rational numbers.
5. We let $A^u \cdot A^v = A^{u+v}$ as long as all expressions exist.
6. We let $(AB)^u = A^u \cdot B^u$ as long as all expressions exist.
7. We let $(A^u)^v = A^{uv} = A^{vu} = (A^v)^u$ as long as all exprressions exist.

Whenever we are working with expressions that contain exponents, it is customary to rewrite them so that all exponents are positive. It is also customary to make any easy simplifications (like Example 1 above).

Example 5 *Rewrite the expression* $3c^{-2}$ *so that all exponents are positive.*

Solution. The key here is knowing how much of the expression is affected by the exponent. We adopt the following convention when working with exponents:

- *An exponent only affects the number or variable that is immediately to its left.*

Since no parentheses are involved in our expression, we assume the exponent only affects the variable c. Therefore, we have

$$3c^{-2} = 3 \cdot \left(\frac{1}{c^2}\right) = \frac{3}{c^2}$$

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Example 6 *Rewrite the expression* $\dfrac{-2t^{-3}}{u^{-1}}$ *so that all exponents are positive.*

Solution. First, note that the exponent -3 does not affect the number -2; it only affects the variable t. When trying to rewrite expressions that are products and quotients of powers, you may find it helpful to break the expression down into a product of fractions, where each variable and constant has its own "column." For example,

$$\frac{-2t^{-3}}{u^{-1}} = \underbrace{\left(-\frac{2}{1}\right)}_{\text{constant column}} \cdot \underbrace{\left(\frac{t^{-3}}{1}\right)}_{t \text{ column}} \cdot \underbrace{\left(\frac{1}{u^{-1}}\right)}_{u \text{ column}}$$

This method isolates each variable and constant and can make dealing with powers easier. Now, we just apply the definition for negative powers, use the rules for dividing fractions, and recombine the columns.

$$\begin{aligned}
\frac{-2t^{-3}}{u^{-1}} &= \left(-\frac{2}{1}\right) \cdot \left(\frac{t^{-3}}{1}\right) \cdot \left(\frac{1}{u^{-1}}\right) \\
&= \left(-\frac{2}{1}\right) \cdot \left(\frac{1}{t^3}\right) \cdot \left(\frac{1}{\frac{1}{u}}\right) \\
&= \left(-\frac{2}{1}\right) \cdot \left(\frac{1}{t^3}\right) \cdot (u) \\
&= -\frac{2u}{t^3}
\end{aligned}$$

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Example 7 *Rewrite the expression* $\dfrac{-5^{-2}9^{3/2}x^{-2/3}}{v^3}$ *so that all exponents are positive.*

Solution. Once again, we will break up this expression into columns and work with the exponents in each column. Observe

$$\begin{aligned}
\frac{-5^{-2}9^{3/2}x^{-2/3}}{v^3} &= \left(-\frac{5^{-2}}{1}\right) \cdot \left(\frac{9^{3/2}}{1}\right) \cdot \left(\frac{x^{-2/3}}{1}\right) \cdot \left(\frac{1}{v^3}\right) \\
&= \left(-\frac{1}{5^2}\right) \cdot \left(\frac{9^{3/2}}{1}\right) \cdot \left(\frac{1}{x^{2/3}}\right) \cdot \left(\frac{1}{v^3}\right) \\
&= \left(-\frac{1}{25}\right) \cdot \left(\frac{\left[\sqrt{9}\right]^3}{1}\right) \cdot \left(\frac{1}{x^{2/3}}\right) \cdot \left(\frac{1}{v^3}\right) \\
&= \left(-\frac{1}{25}\right) \cdot \left(\frac{27}{1}\right) \cdot \left(\frac{1}{x^{2/3}}\right) \cdot \left(\frac{1}{v^3}\right) \\
&= -\frac{27}{25x^{2/3}v^3}
\end{aligned}$$

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Example 8 *Rewrite the expression* $\dfrac{3\left(x^5\right)^{-3}}{\sqrt{z^4}}$ *so that all exponents are positive and in lowest terms.*

Solution. We begin by rewriting the expression so that all exponents are positive and then applying the Laws of Exponents to "clean up" the powers

$$\begin{aligned}
\frac{3\left(x^5\right)^{-3}}{\sqrt{z^4}} &= \left(\frac{3}{1}\right)\cdot\left(\frac{\left[x^5\right]^{-3}}{1}\right)\cdot\left(\frac{1}{\sqrt{z^4}}\right) \\
&= \left(\frac{3}{1}\right)\cdot\left(\frac{1}{\left[x^5\right]^{3}}\right)\cdot\left(\frac{1}{\sqrt{z^4}}\right) \\
&= \left(\frac{3}{1}\right)\cdot\left(\frac{1}{\left[x^5\right]^{3}}\right)\cdot\left(\frac{1}{\left[z^4\right]^{1/2}}\right) \\
&= \left(\frac{3}{1}\right)\cdot\left(\frac{1}{x^{5\cdot 3}}\right)\cdot\left(\frac{1}{z^{4\cdot(1/2)}}\right) \qquad \text{Using } \left(A^u\right)^v = A^{uv} \\
&= \left(\frac{3}{1}\right)\cdot\left(\frac{1}{x^{15}}\right)\cdot\left(\frac{1}{z^{2}}\right) \\
&= \frac{3}{x^{15}z^2}
\end{aligned}$$

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Example 9 *Rewrite the expressions* $\sqrt[3]{x^3}$ *and* $\sqrt{x^2}$ *so that the powers are in lowest terms.*

Solution. This seems like a really easy problem. If we apply the Laws of Exponents the way we did in the previous example, we have

$$\sqrt[3]{x^3} = \left(x^3\right)^{1/3} = x^{3\cdot\frac{1}{3}} = x^1 = x \qquad \sqrt{x^2} = \left(x^2\right)^{1/2} = x^{2\cdot\frac{1}{2}} = x^1 = x$$

However, there is a very subtle problem. Consider $x = -1$. Let's perform the actual computations and compare results.

$$\sqrt[3]{(-1)^3} = \left((-1)^3\right)^{1/3} = (-1)^{1/3} = -1 \qquad \text{[Same anser]}$$

$$\sqrt{(-1)^2} = \left((-1)^2\right)^{1/2} = (1)^{1/2} = 1 \qquad \text{[Not the same answer]}$$

Notice that $\sqrt{x^2}$ gave us the *absolute value* of x when we let $x = -1$. Why did the Laws of Exponents not work? The problem is in the "fine print" of the Laws of Exponents. We are told that

$$\left(A^m\right)^n = A^{mn} = A^{nm} = \left(A^n\right)^m$$

as long as all expressions exist. When the Laws of Exponents actually apply to $\sqrt{x^2}$ we must have

$$\left[x^2\right]^{1/2} = \sqrt{x^2} = \left[x^{1/2}\right]^2$$

This is the problem... since $\sqrt{-1}$ does not exist as a real number, this law of exponents does not apply to $\sqrt{x^2}$ when $x = -1$.

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Final Law of Exponents Let A be any expression. If n is an odd integer, then $\sqrt[n]{A^n} = A$, but if n is an even integer, then $\sqrt[n]{A^n} = |A|$ (the absolute value of A).

Exercises for Unit 3.

Use the Laws of Exponents to simplify the following constant expressions. Give exact answers and do not use a calculator.

(1) $(25)^{3/2}$ (2) $4^{5/2}$ (3) $(27)^{4/3}$ (4) $(27)^{-4/3}$

(5) $(64)^{-3/4}$ (6) $8^{6/3}$ (7) $(100)^{-3/2}$ (8) $(36)^{3/4}$

For what values of b (if any) will the following expressions be undefined?

(9) $\sqrt[3]{5b-9}$ (10) $\sqrt{b-\dfrac{3}{4}}$ (11) $2b-(8-3b)^{1/2}$

(12) $\sqrt[4]{6-7b}$ (13) $\dfrac{(2b-5)^{3/5}}{b+1}$ (14) $\dfrac{b-(4b-12)^{3/2}}{2b+7}$

Rewrite the following expressions so that all exponents are positive and in lowest terms.

(15) $4V^{-3}$ (16) $\dfrac{6}{y^{-2}}$ (17) $2\,(7x)^{-1/3}$ (18) $\dfrac{4^{-1/3}\sqrt{x^2}}{y^5}$

(19) $\dfrac{u^{-3}\sqrt[5]{v^5}}{x^{-2}}$ (20) $\dfrac{2^5\sqrt[4]{A^4}}{t^2\left(\sqrt{u}\right)^5}$ (21) $\dfrac{8^{5/3}6^{-2}}{x^{-3/7}\left(y^{-3}\right)^4}$ (22) $\dfrac{\sqrt[6]{W^{12}}}{\left(3v^2\right)^{-3}}$

(23) Use the fact that $\dfrac{3}{2}=1+\dfrac{1}{2}$ and the Laws of Exponents to explain why $A\sqrt{A}=A^{3/2}$ whenever all expressions exist.

(24) Assuming the expression is defined, is it okay to write $\left(\sqrt{x}\right)^2=x$, or do we need to write $\left(\sqrt{x}\right)^2=|x|$? Justify your answer.

(25) Use the Laws of Exponents to explain why $\sqrt{A^6}=\left|A^3\right|$ whenever all expressions exist.

(26) In Example 5, we wrote $\sqrt{z^4}=z^2$. Why did we not have to use absolute value?

(27) Use the Laws of Exponents to explain why $\sqrt{A^8}=A^4$ whenever all expressions exist. Why is no absolute value needed here?

(28) Write the expression $\sqrt[4]{16x^{12}}$ so that the exponents are in lowest terms. If you need to use absolute value, explain why.