ALGEBRA UNIT 6 — Working with Fractions

In Unit 3, we introduced exponents and showed how they can be used to represent repeated multiplication, taking roots, and division. We also discussed one step involved in *simplifying* an expression — namely rewriting an expression that involves only products and quotients so that all exponents are positive and in lowest terms. This is the first step in the processes required to work effectively with expressions that involve fractions. In this unit, we will explore the other steps. This unit will require all of the units that have come before it.

We say that an expression made up of products and quotients is in *lowest terms* when the following conditions are met:

- All exponents are positive and in lowest terms.
- All repitition of constants and variables has been eliminated.

Example 1 *Place the expression* $\dfrac{uv^{-1}u^3}{vu^5}$ *in lowest terms.*

Solution. As in the Unit 3, we will break this expression up into columns, address the exponents, then "clean up" powers and repititions. Observe

$$\begin{aligned}
\frac{uv^{-1}u^3}{vu^5} &= \left(\frac{u\cdot u^3}{u^5}\right)\cdot\left(\frac{v^{-1}}{v}\right) && \text{Align powers of each variable}\\
&= \left(\frac{u^4}{u^5}\right)\cdot\left(\frac{1}{v\cdot v}\right) && \text{Rewrite so all exponents are positive}\\
&= \left(\frac{1}{u^5}\right)\cdot\left(\frac{1}{v^2}\right) && \text{Combine powers using rules for exponents}\\
&= \frac{1}{u^5v^2} && \text{Recombine variable columns}
\end{aligned}$$

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Example 2 *Place the expression* $\dfrac{3\,(81)^{-3/4}\,xt^5}{x^2t^6}$ *in lowest terms.*

Solution. Observe

$$\begin{aligned}
\frac{3\,(81)^{-3/4}\,xt^5}{x^2t} &= \left(\frac{3}{1}\right)\cdot\left(\frac{81^{-3/4}}{1}\right)\cdot\left(\frac{x}{x^2}\right)\cdot\left(\frac{t^5}{t^6}\right) && \text{Align powers of each variable and constant}\\
&= \left(\frac{3}{1}\right)\cdot\left(\frac{1}{81^{3/4}}\right)\cdot\left(\frac{x}{x^2}\right)\cdot\left(\frac{t^5}{t^6}\right) && \text{Rewrite so all exponents are positive}\\
&= \left(\frac{3}{1}\right)\cdot\left(\frac{1}{\left[\sqrt[4]{81}\right]^3}\right)\cdot\left(\frac{x}{x^2}\right)\cdot\left(\frac{t^5}{t^6}\right) && \text{Simplify constant expression}\\
&= \left(\frac{3}{1}\right)\cdot\left(\frac{1}{27}\right)\cdot\left(\frac{1}{x}\right)\cdot\left(\frac{1}{t}\right) && \text{Combine powers using rules for exponents}\\
&= \frac{1}{9xt}
\end{aligned}$$

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Example 3 *Place the expression* $\dfrac{zy^2z^{3/4}}{4x^{-1}y}$ *in lowest terms.*

Solution. Observe

$$\begin{aligned}
\frac{zy^2z^{3/4}}{4x^{-1}y} &= \left(\frac{1}{4}\right)\cdot\left(\frac{z^1\cdot z^{3/4}}{1}\right)\cdot\left(\frac{y^2}{y}\right)\cdot\left(\frac{1}{x^{-1}}\right) && \text{Align powers of each variable and constant}\\
&= \left(\frac{1}{4}\right)\cdot\left(\frac{z^1\cdot z^{3/4}}{1}\right)\cdot\left(\frac{y^2}{y}\right)\cdot\left(\frac{1}{\frac{1}{x}}\right) && \text{Rewrite so all exponents are positive}\\
&= \left(\frac{1}{4}\right)\cdot\left(\frac{z^{1+3/4}}{1}\right)\cdot\left(y^{2-1}\right)\cdot(x) \qquad (\ y\neq 0) && \text{Combine powers; exclude } y=0\\
&= \frac{z^{7/4}yx}{4} \qquad (\ y\neq 0)
\end{aligned}$$

Notice that we placed the restriction that y be nonzero in the step where we combined powers. We have to do this, because the original expression is not defined when $y=0$, while the expression we get by combining powers actually is defined when $y=0$. Therefore, if we want these two expressions to be equal, we must remove the possibility that $y=0$ from that step, and from every step that follows.

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Exclusion Rule for Reducing: When you are placing an expression A in lowest terms and a variable expression B completely cancels from the denominator of A in one of the steps, you have to reqiure that B be nonzero in that step and every successive step of process.

Example 4 *Solve the equation* $\dfrac{x}{(y-1)^2}=\dfrac{z}{y-1}$ *for the variable* y*. Use the Exclusion Rule when necessary.*

Solution. Solving for a variable in an equation always requires us to get all occurrances of that variable out of the denominators of the terms making up the equation. If the left and right hand expressions in the equation consist of a single term, we first place both sides of the equation in lowest terms, and then multiply both sides by any factors from the denominator that contain the variable we wish to solve for. In this case, observe that both expressions are already in lowest terms. Now,

$$\begin{aligned}
\frac{x}{(y-1)^2}=\frac{z}{y-1} &\implies \left[\frac{x}{(y-1)^2}\right](y-1)^2=\left[\frac{z}{y-1}\right](y-1)^2 && \text{Multiply both sides by } (y-1)^2\\
&\implies x=\left(\frac{z}{1}\right)\left(\frac{(y-1)^2}{y-1}\right) && \text{Apply Cancellation Law; group powers}\\
&\implies x=z(y-1) \qquad (y\neq 1) && \text{Apply Exclusion Rule}\\
&\implies \frac{x}{z}=\frac{z(y-1)}{z} \qquad (y\neq 1) && \text{Divide both sides by } z\\
&\implies \frac{x}{z}=y-1 \qquad (y\neq 1, z\neq 0) && \text{Apply Cancellation Law; exclude } z=0\\
&\implies \frac{x}{z}+1=(y-1)+1 \qquad (y\neq 1, z\neq 0) && \text{Add 1 to both sides}\\
&\implies \frac{x}{z}+1=y \qquad (y\neq 1, z\neq 0) && \text{Combine like terms}
\end{aligned}$$

In the solution process, we excluded $y=1$ because the original equation *is not* defined when $y=1$ but the final solution *is* defined when $y=1$. We had to exclude $z=0$ along the way as well because the original equation *is* defined when $z=0$ but the final solution *is not.*

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Example 5 *Solve the equation* $\dfrac{xy^{-4}}{x^2} = \dfrac{z^2y}{y^3x}$ *for the variable* z. *Use the Exclusion Rule when necessary.*

Solution. First, we will rewrite the expression on each side of the equation in lowest terms. This makes the expressions easier to work with, and it helps us decide if the Exclusion Rule will be needed.

$$\text{Left Expression:} \qquad \begin{aligned} \frac{xy^{-4}}{x^2} &= \left(\frac{x}{x^2}\right)\cdot\left(\frac{y^{-4}}{1}\right) \\ &= \left(\frac{1}{x}\right)\cdot\left(\frac{1}{y^4}\right) \\ &= \frac{1}{xy^4} \end{aligned}$$

$$\text{Right Expression:} \qquad \begin{aligned} \frac{z^2y}{y^3x} &= \left(\frac{1}{x}\right)\cdot\left(\frac{y}{y^3}\right)\cdot\left(\frac{z^2}{1}\right) \\ &= \left(\frac{1}{x}\right)\cdot\left(\frac{1}{y^2}\right)\cdot\left(\frac{z^2}{1}\right) \\ &= \frac{z^2}{xy^2} \end{aligned}$$

We are now ready to solve the equation for the variable z. Observe

$$\begin{aligned} \frac{xy^{-4}}{x^2} = \frac{z^2y}{y^3x} \quad &\Longrightarrow \quad \frac{1}{xy^4} = \frac{z^2}{xy^2} && \text{Substitute reduced expressions on both sides} \\ &\Longrightarrow \quad \left(\frac{1}{xy^4}\right)(xy^2) = \left(\frac{z^2}{xy^2}\right)(xy^2) && \text{Multiply both sides by } xy^2 \\ &\Longrightarrow \quad \frac{xy^2}{xy^4} = z^2 && \text{Apply Cancellation Law} \\ &\Longrightarrow \quad \left(\frac{x}{x}\right)\cdot\left(\frac{y^2}{y^4}\right) = z^2 && \\ &\Longrightarrow \quad \frac{1}{y^2} = z^2 \qquad (x \neq 0) && \text{Apply Exclusion Rule for } x \\ &\Longrightarrow \quad \pm\sqrt{\frac{1}{y^2}} = z \qquad (x \neq 0) && \\ &\Longrightarrow \quad \pm\frac{\sqrt{1}}{\sqrt{y^2}} = z \qquad (x \neq 0) && \text{Use Exponent Rule } \left(\frac{a}{b}\right)^u = \frac{a^u}{b^u} \\ &\Longrightarrow \quad \pm\frac{1}{|y|} = z \qquad (x \neq 0) && \end{aligned}$$

We could also exclude $y = 0$ from the final solution. There is nothing wrong with doing this; however, since the original equation and the final solution are *both* undefined when $y = 0$ (as is every step along the way in the solution process), we don't have to.

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We complete this unit with a discussion on how to combine fractions. This is an important skill that is required whenever you need to solve equations involving unknowns that are in the denominator of one or more fractions. The underlying rule is the same as the one for combining fractions of real numbers.

$$\text{If } A, B, C, D \text{ are expressions and } B,\ D \text{ are not 0, then } \quad \frac{A}{B} + \frac{C}{D} = \frac{AD + BC}{BD}$$

This rather complicaated-looking rule actually comes from the Distributive Law, but we will simply take it on faith.

When confronted with an expression that consists of a sum of fractions, it is customary to rewrite it as a single fraction using the rule above. Once that is taken care of, it is also common practice to place the resulting fraction in lowest terms.

Example 6 *Rewrite the expression* $\dfrac{3}{2}+\dfrac{4x}{y}$ *as a single fraction.*

Solution. Applying the rule for combining fractions, we see that

$$\begin{aligned}\frac{3}{2}+\frac{4x}{y} &= \frac{3\cdot y+2\cdot(4x)}{2y}\\ &= \frac{3y+8x}{2y}\end{aligned}$$

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You might be tempted to cancel the y from the numerator and denominator, but this would not be correct. Remember, canceling expressions can only be done if the expressions are factors of the numerator and the denominator (which means they can be isolated in a "column" like we did in Unit 3).

Cancelling Expressions in Fractions that Contain Sums

- You can cancel an expression from a fraction only when it appears as a factor in *every* term in the numerator and *every* term of the denominator.

In other words, you may cancel an expression from a fraction only if the Cancellation Law can be applied. For example, the variable t cannot be cancelled from any of the following expressions.

$$\frac{1+t}{t} \qquad \frac{t^2-1}{t+1} \qquad \frac{t^3+\sqrt{t}}{1-t}$$

because it is not possible to isolate t as a factor of *each* term in the numerator and denominator. On the other hand, it is possible to cancel the variable t from the expression

$$\frac{t^3+2t}{4t\sqrt{t}}$$

To see why, observe

$$\begin{aligned}\frac{t^3+2t}{4t\sqrt{t}} &= \frac{(t)(t^2)+(t)(2)}{(t)(4\sqrt{t})} && \text{Isolate } t \text{ as a factor of } \textit{each} \text{ term}\\ &= \frac{t\left(t^2+2\right)}{t\left(4\sqrt{t}\right)} && \text{Pull } t \text{ from each term}\\ &= \left(\frac{t}{t}\right)\cdot\left(\frac{t^2+2}{4\sqrt{t}}\right) && \text{Align the pulled } t\text{'s in a column}\\ &= \frac{t^2+2}{4\sqrt{t}} && \text{Apply Cancellation Law}\end{aligned}$$

We could add the restriction that $t\neq 0$ since t cancelled from the denominator. However, the expression $\sqrt{t}$ remains as a factor of the denominator, and so the simplified expression remains undefined when $t=0$. Consequently, it is not necessary to apply the Exclusion Rule (although it is not wrong either).

Example 7 *Solve the equation* $\frac{3}{a} + 2t = 1$ *for the variable* a.

Solution. When trying to solve an equation for a variable that appears in the denominator of a fraction, the goal is to use the following three step process:

1. Use the rule for combining fractions to rewrite each side of the expression as a single fraction.
2. Multiply both sides of the equation by the denominator that contains the variable we want to solve for and use the Cancellation Law to "remove" the variable from the denominator.
3. Solve the new equation using the processes from earlier units.

Let's see how this process evolves for the equation we are considering. Observe

$$
\begin{array}{lll}
\frac{3}{a} + \frac{2t}{1} = 1 & \Longrightarrow \frac{3(1) + (2t)(a)}{(a)(1)} = 1 & \text{Rewrite left side as one fraction} \\
& \Longrightarrow \frac{3 + 2ta}{a} = 1 & \\
& \Longrightarrow \left(\frac{3 + 2ta}{a}\right)(a) = (1)(a) & \text{Multiply both sides by } a \\
& \Longrightarrow 3 + 2ta = a \qquad (a \neq 0) & \text{Apply Cancellation Law; exclude } a = 0 \\
& \Longrightarrow (3 + 2ta) - 2ta = a - 2ta \qquad (a \neq 0) & \text{Subtract } 2ta \text{ from both sides} \\
& \Longrightarrow 3 = (1 - 2t)a \qquad (a \neq 0) & \text{Combine like terms} \\
& \Longrightarrow \frac{3}{1 - 2t} = \frac{(1 - 2t)a}{1 - 2t} \qquad (a \neq 0) & \text{Divide both sides by } 1 - 2t \\
& \Longrightarrow \frac{3}{1 - 2t} = a \qquad (a \neq 0, \quad t \neq 1/2) & \text{Apply Cancellation Law; exclude } t = 1/2
\end{array}
$$

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Example 8 *Solve the equation* $\frac{3}{x} - \frac{x}{3} = 1$ *for the variable* x.

$$
\begin{array}{lll}
\frac{3}{x} - \frac{x}{3} = 1 & \Longrightarrow \frac{3(3) - x(x)}{3x} = 1 & \text{Rewrite left side as single fraction} \\
& \Longrightarrow \frac{9 - x^2}{3x} = 1 & \\
& \Longrightarrow \left(\frac{9 - x^2}{3x}\right)(3x) = 1(3x) & \text{Multiply both sides by } 3x \\
& \Longrightarrow 9 - x^2 = 3x & \text{Apply Cancellation Law}
\end{array}
$$

At this stage, we have rewritten the original equation as a quadratic equation with respect to the variable x. We can now solve this equation using the Quadratic Formula (or completing the square).

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\begin{array}{lll}
9 - x^2 = 3x & \Longrightarrow 0 = x^2 + 3x - 9 & \text{Move all terms to the right side} \\
& \Longrightarrow x = -\frac{-3}{2(1)} \pm \frac{\sqrt{(-3)^2 - 4(-1)(9)}}{2(1)} & \text{Apply quadratic formula} \\
& \Longrightarrow x = -\frac{3}{2} \pm \frac{\sqrt{45}}{2} & \\
& \Longrightarrow x = -\frac{3}{2} \pm \frac{3\sqrt{5}}{2} &
\end{array}
$$

When we solved the equation $9 - x^2 = 3x$ in the last example, we started by moving all terms to the right-hand side of the equation. We could also have moved all terms to the left-hand side and proceeded with the quadratic formula. There is a slight advantage to moving all terms to the right-hand side in this case — the coefficient of x^2 is positive. It is not necessary for the coefficient of x^2 to be positive, but it does make applying the quadratic formula a little easier. (Try it the other way and see what the difficulty is.)

Exercises for Unit 6

In each expression below, cancel the highest power of x possible from the numerator and denominator. In some cases, it may not be possible to cancel any power of x. Apply the Exclusion Rule when needed.

$$(1)\ \frac{2x+x^2}{x} \qquad (2)\ \frac{1+x}{x-1} \qquad (3)\ \frac{2+x\sqrt{x}}{y-2x}$$

$$(4)\ \frac{x^3-tx^2}{x^4-6x^2} \qquad (5)\ \frac{px^{1/2}+\pi\sqrt{x}}{\sqrt{x}} \qquad (6)\ \frac{3x^{3/2}+ax}{bx^2}$$

Place each of the following expressions in lowest terms. Apply the Exclusion Rule when needed.

$$(7)\ 8^{-4/3}u^5u^{-2} \qquad (8)\ \frac{3c^2b^{-2}}{c^{-3}b} \qquad (9)\ \frac{x^{3/2}y^3x}{ty} \qquad (10)\ 4\left(h^3\right)^{3/2}z^{-2}z^4$$

$$(11)\ \frac{\sqrt[3]{s^3}r^{3/4}}{rs} \qquad (12)\ \frac{\left(3x^3\right)^{-2}x}{z\sqrt{z}} \qquad (13)\ \frac{5\left(5v^{4/3}\right)^{-3}xy}{v} \qquad (14)\ \frac{\sqrt[4]{uv^4}u}{3u^3}$$

$$(15)\ \left(2x^{-2}y\right)^{3/2}x \qquad (16)\ \frac{6g^3\sqrt{g^4}}{g^{-1}} \qquad (17)\ \frac{h^{3/2}k^3k^{1/2}}{hk^3j} \qquad (18)\ \sqrt{\frac{9w^8}{z^6}}$$

19 Solve the equation $\dfrac{2}{a}+1=3$ for the variable a.

20 Solve the equation $\dfrac{3}{2}+y=\dfrac{1}{x}$ for the variable x.

21 Solve the equation $\dfrac{y+1}{xy}=6$ for the variable y.

22 Solve the equation $2-\dfrac{pt}{t-1}=3$ for the variable t.

23 Solve the equation $\dfrac{4}{b}+\dfrac{1}{b+1}=1$ for the variable b.

24 Solve the equation $\dfrac{2u}{1-u}+u=0$ for the variable u.

25 Solve the equation $\dfrac{w^2}{1-w}+1=w-1$ for the variable w.