FUNCTION COMPOSITION

The Definition

Suppose that f and g are functions, and suppose that the *outputs* of the function g all lie in the domain of the function f.

We can create a new function whose *inputs* come from the domain of g but whose *outputs* come from f.

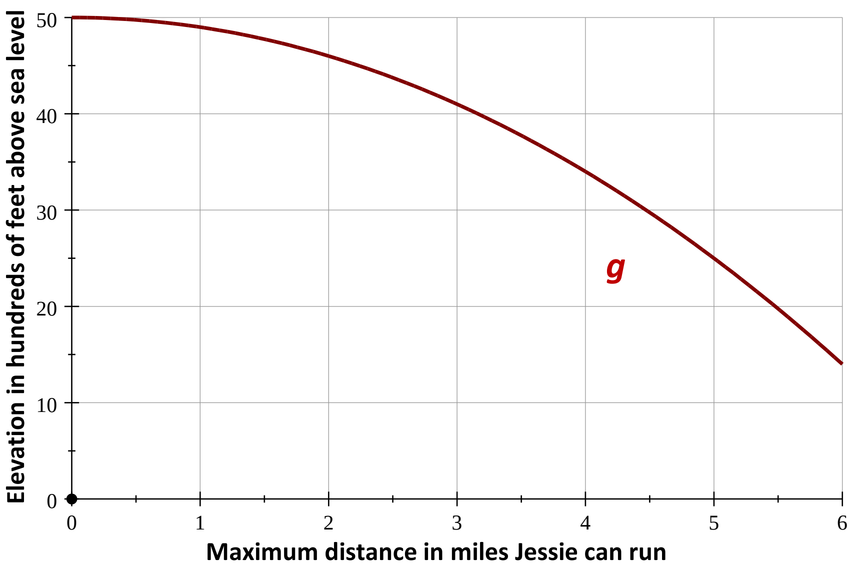
This new function is called the ***composition of f after g***. It is denoted by the special symbol $f \circ g$.

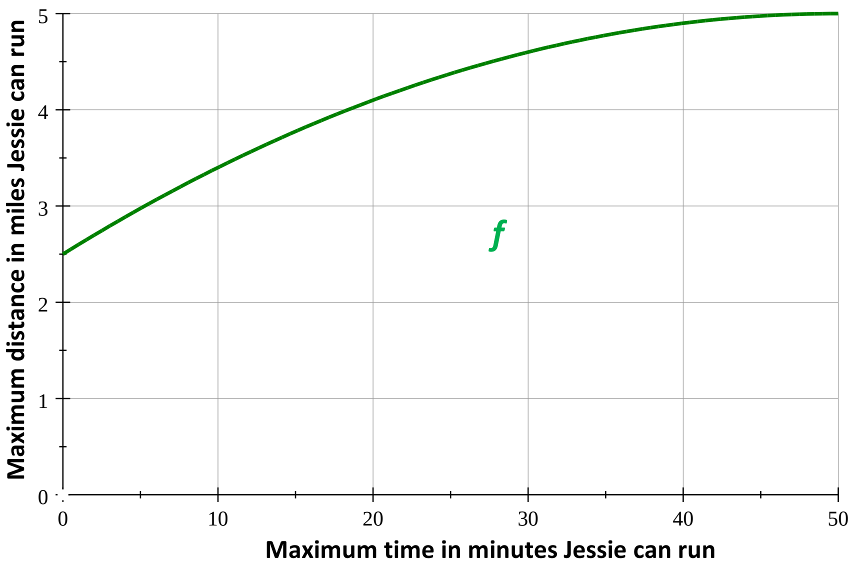
The *output* of the function $f \circ g$ is denoted by $[f \circ g](x)$. This output comes from taking the output of g and using it as input into f. In symbols,

$$[f \circ g](x) = \underbrace{f(g(x))}$$

The output of $f \circ g$ is created by taking the *output* of g as *input* into f.

Jessie likes to run while he is on business trips. The graphs below provide some information about Jessie's running.





The graph on the left gives Jessie's elevation as a function of the number of miles he is able to run. We will call this function *g*.

The graph on the right gives the number of miles Jessie is able to run as a function of the time in minutes that he can run. We will call this function *f*.

- When Jessie can run for twenty-five minutes, what is his approximate elevation? Use function notation to express your answer.

We will answer this question using a two-step process. First, notice that if we *input* 20 minutes into the function *f*, we find that Jessie can run a maximum distance of about 4.4 miles. In function notation, we have

$$f(25) \approx 4.4 \text{ miles}$$

This is important, because the function *g* gives Jessie's elevation in terms of the maximum distance he can run. Looking at the graph for *g*, we see that

$$g(4.4) \approx 33 \text{ hundred feet}$$

Thinking purely in terms of functions, we see that when we *input* 20 minutes into *f*, the *output* is 4.4 miles; and when we *input* 4.4 miles into *g*, the *output* is 33 hundred feet.

We can use function composition to summarize this two-step process:

$$[g \circ f](25) = g\big(f(25)\big) = g(4.4) = 33$$

Step 1: Input 25 into *f*

Step 2: Use the *output* as input into *g*.

- What is the meaning of the expression $[g \circ f](15)$ and what is the value?

Note that $[g \circ f](15) = g(f(15))$. We can understand the meaning of this expression in several ways.

INPUT-OUTPUT Understanding
The expression $[g \circ f](15)$ is the *output* from *g* obtained when $f(15)$ is used as *input* into *g*.

TWO-STEP Understanding
The expression $[g \circ f](15)$ is the end result of a two-step process

1. Input 15 minutes into the rule we call *f*.
2. Take the resulting output from f, and input this value into the rule we call *g*.

PRACTICAL Understanding
The expression $[g \circ f](15)$ is Jessie's elevation in hundreds of feet when he is able to run a maximum of 15 minutes.

The right-hand graph tells us $f(15) \approx 3.75$ miles, and the left-hand graph tells us $g(3.75) \approx 36$ hundred feet. Therefore

$$[g \circ f](15) = g\big(f(15)\big) \approx g(3.75) \approx 36 \text{ hundred feet}$$

- Is there a practical meaning for the expression $[f \circ g](5)$?

The expression $[f \circ g](5)$ has no practical meaning *in the context of this problem*.

This is because $[f \circ g](5) = f\big(g(5)\big)$. The input into the rule we call f must be a time in minutes that Jessie can run. However, $g(5)$ is an elevation in hundreds of feet above sea level. Consequently, $g(5)$ is not a legitimate input into f.

If functions arise in a context, it is critical to keep this context in mind. The expression

$$[f \circ g](5)$$

does not make sense in the context of Jessie's running.

However, if we temporarily ignore this context, the expression $[f \circ g](5)$ does have numeric value. Using the graphs without regard to the meanings of the input and output values, we see

$$[f \circ g](5) = f\big(g(5)\big) \approx f(25) \approx 4.4$$

This computation makes no sense in the context of this problem and is INCORRECT.

- Construct the output formula for $f \circ g$ and the output formula for $g \circ f$ if f and g are defined by the output formulas

$$u = f(v) = \frac{1}{1+u} \qquad v = g(y) = \sqrt{y}$$

To create the output formula for $f \circ g$ we simply use the output formula for the function g as input into the output formula for f.

$$[f \circ g](y) = \frac{1}{1 + g(y)} = \frac{1}{1 + \sqrt{y}}$$

To create the output formula for $g \circ f$ we simply use the output formula for the function f as input into the output formula for g.

$$[g \circ f](v) = \sqrt{f(v)} = \sqrt{\frac{1}{1 + v}}$$

Neither of the functions f or g is embedded in a context, so we don't have to be concerned about whether or not $f \circ g$ and $g \circ f$ have practical meaning.

Since these functions have no practical context, the only concerns about meaning come from the domains of the compositions.

- What is the domain for the function $g \circ f$?

We will have division by 0 when $1 + v = 0$. This happens when $v = -1$, so we know that $v = -1$ is not in the domain of $g \circ f$.

We will have the square root of a negative number when $1 + v < 0$. This happens when $v < -1$, so we know that no value $v < -1$ can be in the domain of $g \circ f$.

The domain of the function $g \circ f$ will be the set of inputs $v > -1$.

- What is the domain for the function $f \circ g$?

We do not have to worry about division by 0, because $\sqrt{y}$ is NEVER NEGATIVE.

We will have the square root of a negative number when $y < 0$.

The domain for $f \circ g$ will be the set of inputs $y \geq 0$.