Prime Ideals and the Classical Topology on Graphs

Brian Frazier
Mississippi State University

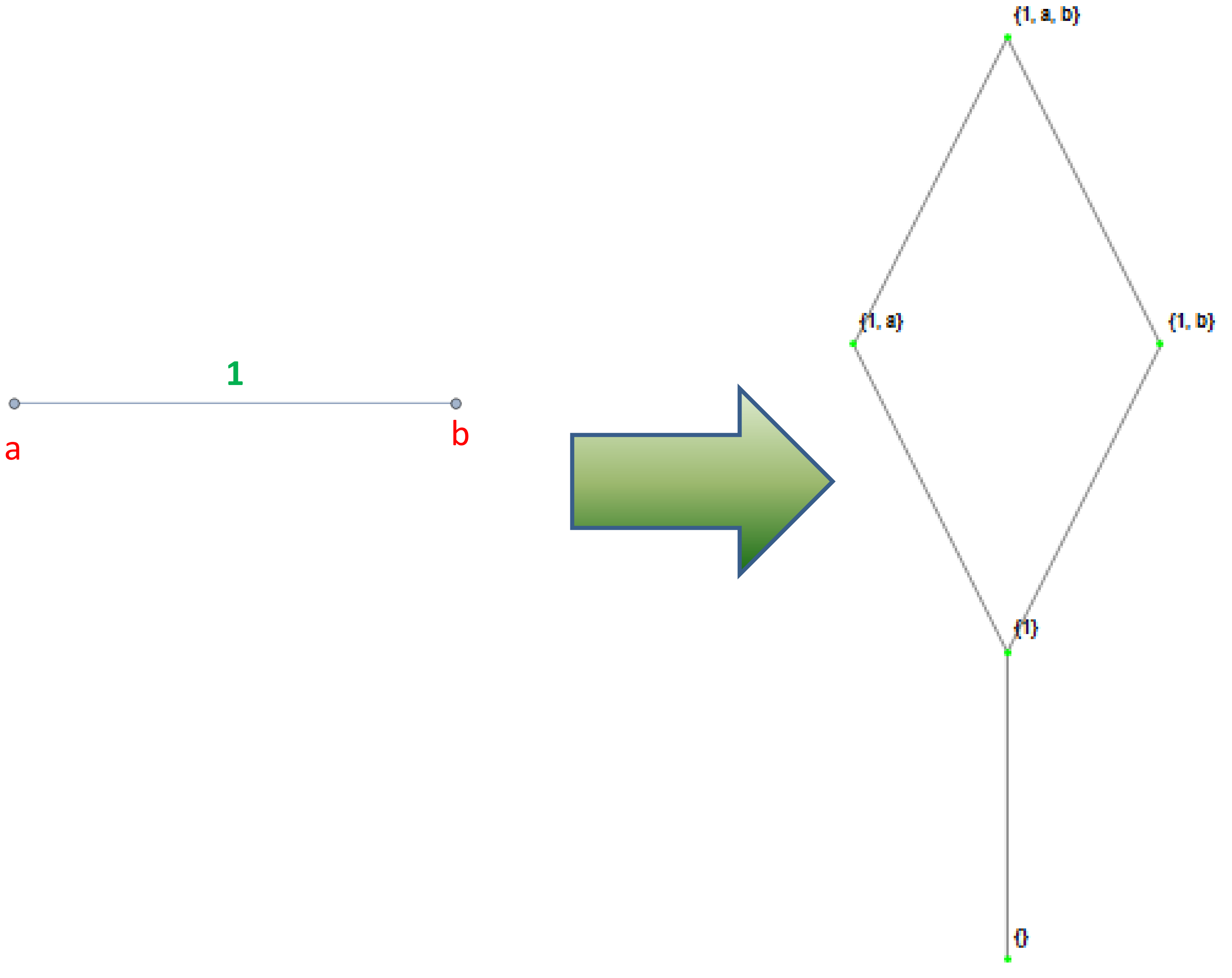
James Hart
Middle Tennessee State University

The “Classical” Topology on Graphs

- Introduced by Antoine Vella in his 2005 PhD dissertation
- Let $\mathbb{G} = (G, E)$ be a graph. The set G consists of the graph vertices, and the set E consists of the graph edges.
- Let $x \in \mathrm{G}$ and let $E(x)$ denote the set of all edges incident to x.
- The classical topology is defined on the set $G \cup E$ as follows

  A set $U \subseteq G \cup E$ is open provided
  ❖ $U \subseteq E$ or
  ❖ $x \in U \cap G$ guarantees $E(x) \subseteq U$

- The classical topology is the coarsest topology under which graph homomorphisms are continuous.
- Let $\Omega(\mathbb{G})$ denote the open-set lattice for a graph $\mathbb{G}$.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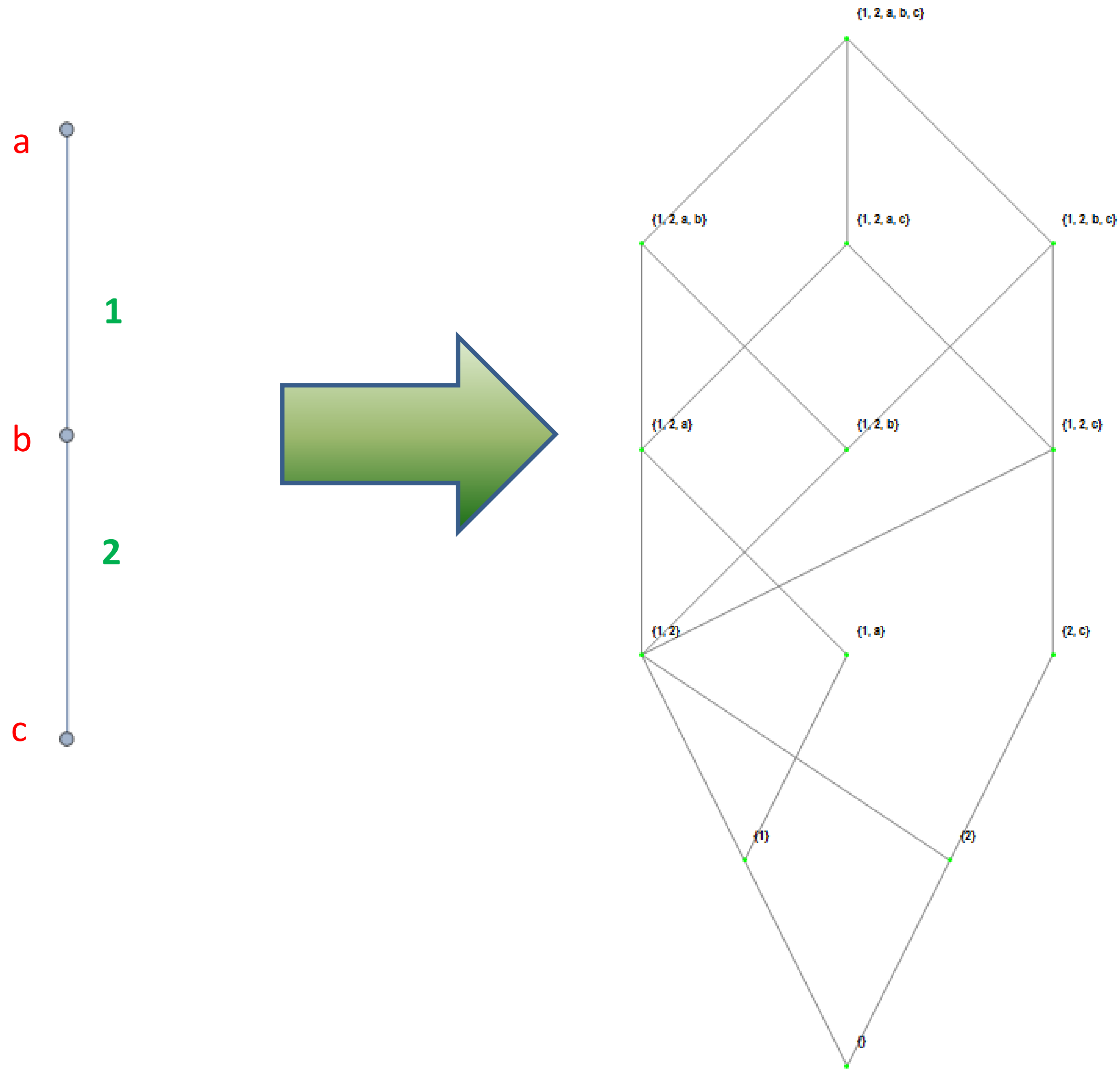
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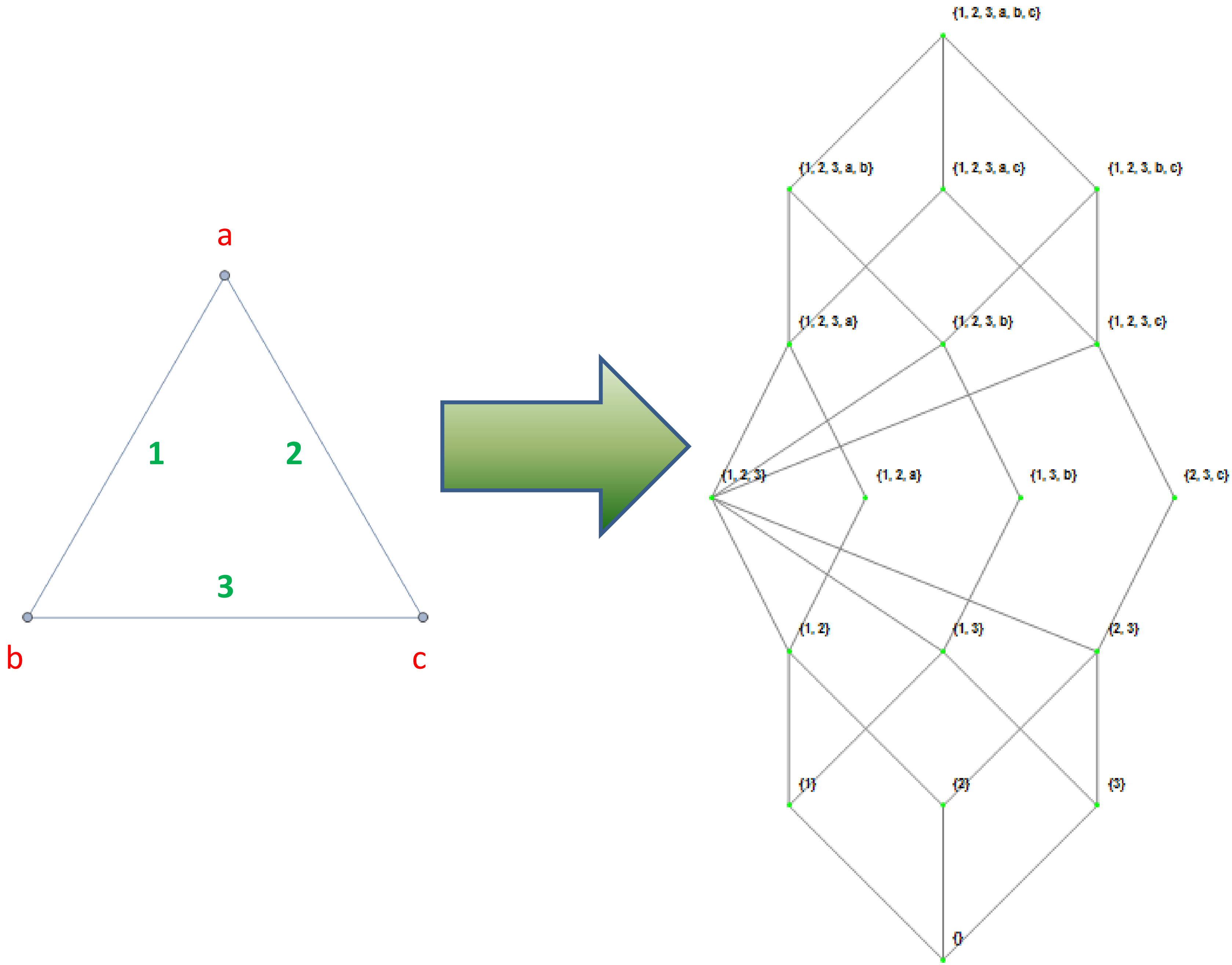
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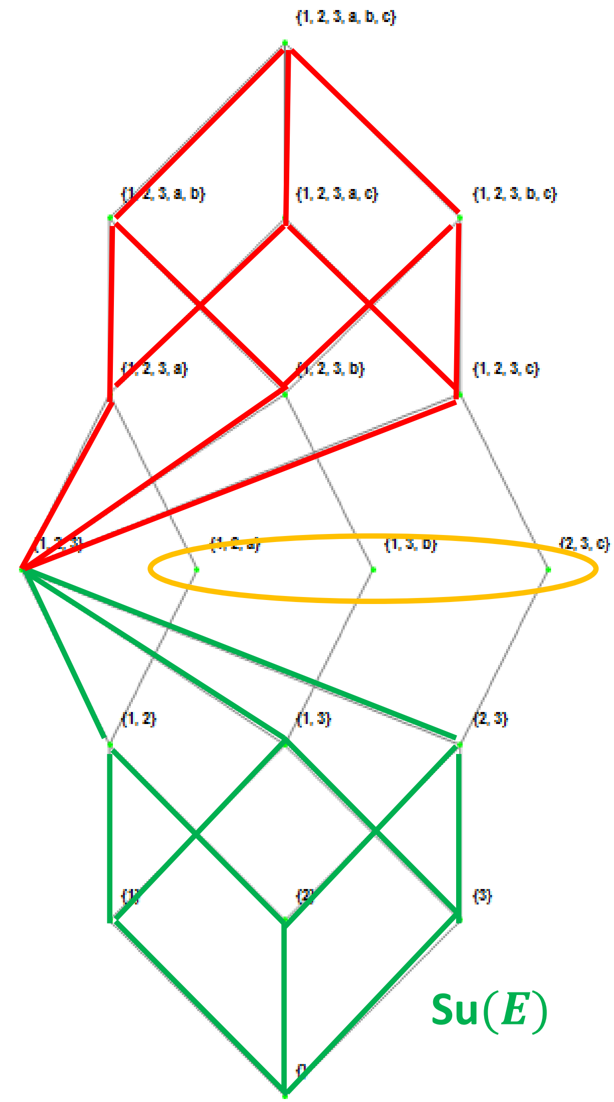
- These lattices looked strangely familiar, but I could not *quite* determine why.
- We began exploring the structure of these lattices.

Much of the lattice is contained in a "Boolean Cone" comprised of Su(E) and a sublattice isomorphic to Su(G).

The sublattice Su(E) contains no useful information about the graph $\mathbb{G}$ since it merely reflects the cardinality of the edge set.

Information about the structure of $\mathbb{G}$ is *probably* contained in the elements "suspended" between the nappes of the cone.



- Momentarily stymied, we turned attention to the prime ideals of these lattices.

- We restricted attention to simple graphs with no isolated vertices to avoid undue complexity.

- The maximal ideals of $\Omega(\mathbb{G})$ are always principal and are generated by those open sets that contain E but are *missing exactly one vertex*. (These are the co-atoms of the upper nappe of the Boolean Cone.)

As long as the graph in question is NOT the graph K_2 the following statements are true.

- A nonmaximal ideal of $\Omega(\mathbb{G})$ is prime if and only if it is generated by an open set that is *missing exactly one edge and the two vertices incident to it*. (These are the maximal "suspended" elements.)

- The intersection of two maximal ideals will contain at most one prime ideal.

- The intersection of two maximal prime ideals will contain a prime ideal if and only if *the vertices missing from the generators are adjacent*. The prime ideal in the intersection is generated by the open missing these vertices and the edge between them.

MAIN RESULT

- A nonmaximal ideal of $\Omega(\mathbb{G})$ is prime if and only if it is generated by an open set that is missing exactly one edge and the two vertices incident to it.

Outline of Proof.

If I is a nonmaximal prime ideal, then $E \notin I$. Therefore, $P = I \cap \mathrm{Su}(E)$ is a prime ideal of $\mathrm{Su}(E)$.

It follows that P is generated by a subset χ of E missing exactly one edge. Let e be the edge missing from χ and let x, y be the vertices incident to e.

Let A be the ideal generated by the open $\mu = \chi \cup (G - \{x, y\})$. Clearly, $I \subseteq A$.

Observe that $\mu \cap E(x) \in I$, but $E(x) \notin I$. (Otherwise, we have $E \in I$.) We must conclude $\mu \in I$. Hence, $I = A$ as desired.

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On the other hand, if μ is an open missing exactly one edge and the two vertices incident to it, then it is easy to see that μ is meet-irreducible in $\Omega(\mathbb{G})$.

Suppose that $\mathbb{G} = (G, E)$ is a simple graph with no isolated vertices, endowed with the classical topology.

Let $\mathrm{MP}[\Omega(\mathbb{G})]$ denote the subposet of meet-prime elements of $\Omega(\mathbb{G})$.

- Every member of $\mathrm{MP}[\Omega(\mathbb{G})]$ that is not maximal is covered by exactly two maximal members.

- There is a bijective correspondence between the set G and the set G' of maximal members of $\mathrm{MP}[\Omega(\mathbb{G})]$.

$$\boldsymbol{f_G(x) = (E \cup G) - \{x\}}$$

- There is a bijective correspondence between the set E and the set E' of minimal members of $\mathrm{MP}[\Omega(\mathbb{G})]$.

$$\boldsymbol{f_E(e) = (E - \{e\}) \cup (V - \{x_e, y_e\})}$$

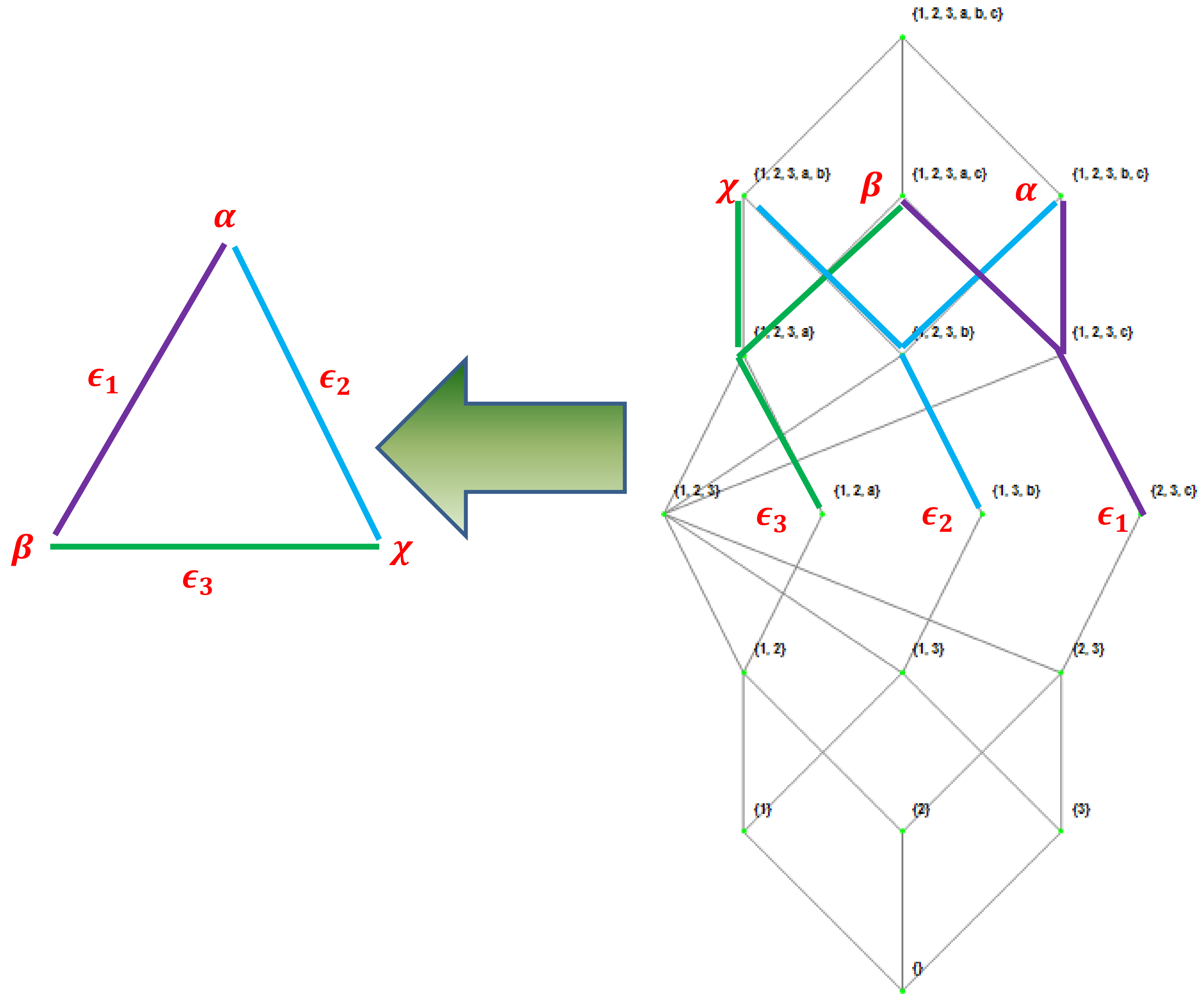
- Two vertices x, y are adjacent via edge e if and only if

$$\boldsymbol{f_E(e) \in f_G(x) \cap f_G(y)}$$

Construct a new graph $\mathbb{G}' = (G', E')$ in the following way:

- Let G' consist of the maximal elements of $\mathrm{MP}[\Omega(\mathbb{G})]$.
- Let E' consist of the minimal elements of $\mathrm{MP}[\Omega(\mathbb{G})]$.
- $\epsilon \in E'$ is incident to $\chi, \upsilon \in G'$ if and only if $\epsilon \subseteq \chi \cap \upsilon$.

The graph $\mathbb{G}$ is graph-isomorphic to the graph $\mathbb{G}'$.


α
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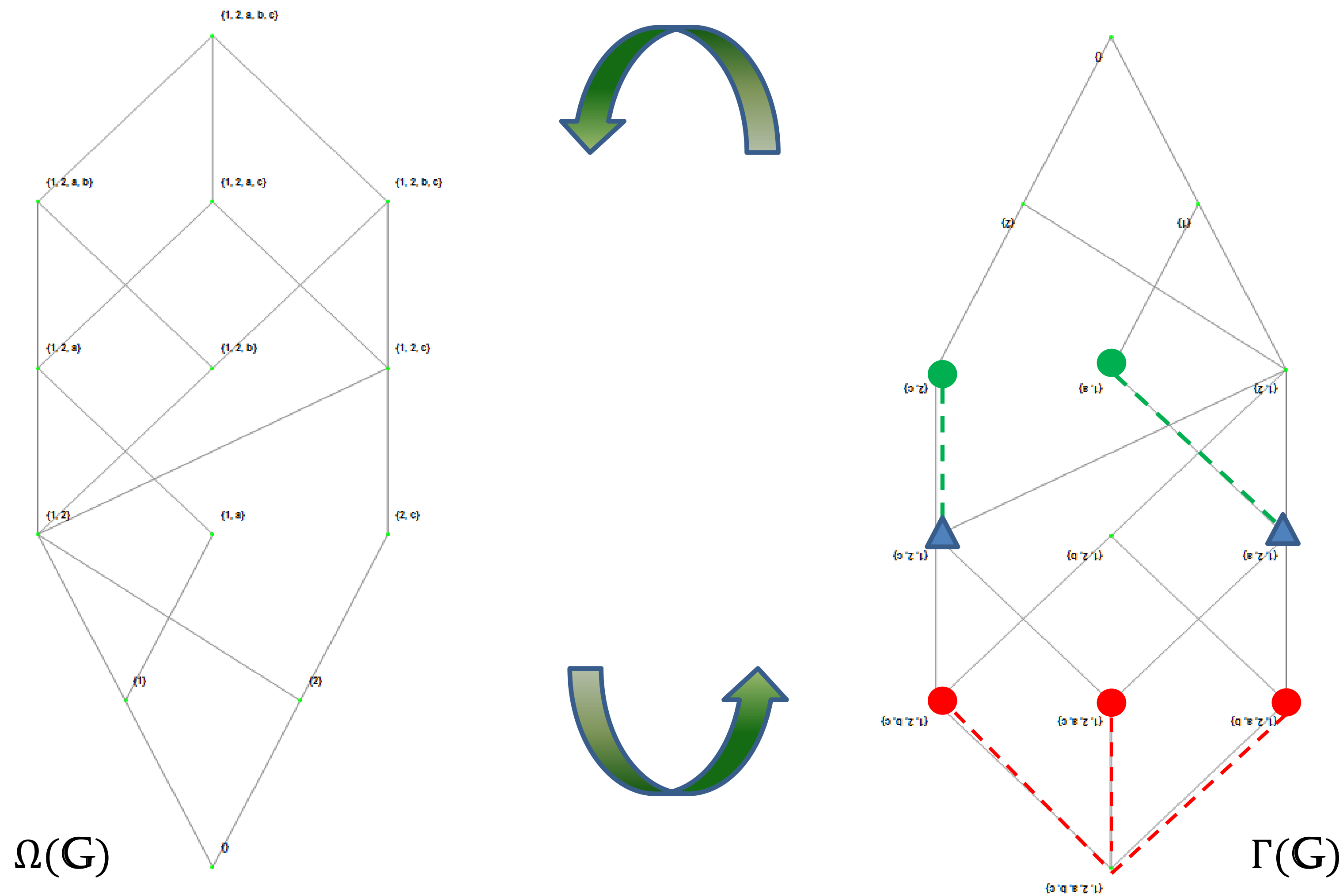
WHERE TO FROM HERE?

- What properties characterize the lattices that are isomorphic to the open set lattices of simple graphs with no isolated vertices?



CONJECTURES

- If there is a maximum degree n for all vertices, then $\Gamma(\mathbb{G})$ is isomorphic to the ideal completion of a relatively n-normal lattice.

- The completely join-prime members of $\Gamma(\mathbb{G})$ comprise a two-level “ordinal decomposition” of the lattice.

- Suppose L is a bialgebraic, distributive lattice whose subposet $\mathrm{MP}[L]$ satisfies the following properties:
  - ❖ Every element of $\mathrm{MP}[L]$ is either maximal or is covered by exactly two maximal members.
  - ❖ Every chain in $\mathrm{MP}[L]$ has length 2.

The lattice L is order-isomorphic to the open set lattice of some hypergraph endowed with the classical topology.