Key to Homework from Module 3

The problems below were assigned as email-homework from Module 3. They were due by 12:00 AM Thursday morning on March 13, 2014.

(4) [p.109] Windows, Inc. manufactures specialty windows. One of their styles is in the shape of a semicircle composed of three pie-shaped panes of glass, each of equal area (see diagram in workbook). Each pane is surrounded by a thin metal frame. Let L be the length of the base of the window, measured in feet from left to right. When a customer orders this style, she specifies the value of L.

(a) Determine the total length of the frame for a window if $L = 4$ feet.
(b) Determine a function h that relates the total length of the frame to the length of the base of the window.
(c) If the cost of the framing material is twelve dollars per linear foot, what is the cost of a window frame with a base length of four feet?
(d) Suppose your budget limits you to spending \$500 on your window frame. What is the longest base length you can afford?

Solution. The total length of the frame will be the length of the base plus the length of the frame separating each panel plus the length of the window circumference. If the base length is $L = 4$, then the radius of the semicircle is two feet. Consequently, the frames separating each panel have length two feet as well, and the circumference of the semicircle will be 2π feet. Thus, the total length of the frame required will be

$$4 + 2 + 2 + 2\pi = 8 + 2\pi \approx 14.28 \text{ feet}$$

For Part (b), let R denote the radius of the semicircle, measured in feet from the center of the circle and let C be the circumference of the semicircle, measured in feet from the left end of the base. If L denotes the length of the base, then $R = L/2$, and

$$C = \pi R = \frac{\pi L}{2}$$

Therefore, the function we want is $h(L) = L + 2\left(\frac{L}{2}\right) + \frac{\pi L}{2}$.

For Part (c), we already know that the total length of the framing is approximately 14.28 feet. Consequently, the cost of the frame will be approximately

$$(14.28 \text{ feet}) \times \left(\frac{12 \text{ dollars}}{1 \text{ foot}}\right) = 171.36 \text{ dollars}$$

To answer Part (d), we first need the *cost function* for the frame. If we let D denote the function that relates the cost in dollars of the frame to its length L in feet, then we know that $D(L) = 12h(L)$. It makes sense that the maximum length *should* occur when we have spent all of our \$500. This means we want to solve the equation $500 = 12h(L)$ for the variable L. It helps to write the formula for h as a single fraction. Doing so, we find

$$h(L) = \frac{L(4+\pi)}{2}$$

Now, we simply solve the the value of L. Observe

$$\begin{aligned}
500 = 12h(L) &\implies 60 = h(L) \\
&\implies 60 = \frac{L(4+\pi)}{2} \\
&\implies 120 = L(4+\pi) \\
&\implies \frac{120}{4+\pi} = L
\end{aligned}$$

Therefore, the maximum length of the frame will be $L \approx 16.8$ feet.

It is worth noting that the problem does not directly ask us to solve the equation $500 = 12h(L)$. It merely asks us to find the *maximum* length we can afford on our budget of five hundred dollars. Does this really happen when we have spent all of our money? The maximum length we can afford *might* occur at any cost in the interval $0 \le D \le 500$, so how can we be sure we have found the true maximum?

We can write the length as a function of the cost D simply by solving $D = 12h(L)$ for the variable L. If we do so, we find

$$\begin{aligned}
D = 12h(L) &\implies \frac{D}{12} = h(L) \\
&\implies \frac{D}{12} = \frac{L(4+\pi)}{2} \\
&\implies \frac{D}{6} = L(4+\pi) \\
&\implies \frac{D}{6(4+\pi)} = L
\end{aligned}$$

Now, we see that L is a *linear* function of D; in fact, it has a constant rate of change $m = \frac{1}{6(4+\pi)}$.

Linear functions do not have turning points in their graphs like parabolas and some other nonlinear functions. Since the constant rate of change is *positive*, we know that the graph for this function is rising from left to right. We must conclude that the largest value for L will occur at the rightmost endpoint of the relevant domain for this function. Since our budget has been limited to \$500, we know that the relevant domain is $0 \le D \le 500$. Thus, the maximum value of L really must occur when $D = 500$.

(24) [p 111] For each of the relations shown in Problem 23, determine whether x is a function of y. Justify your answers. (See workbook for graphs and tables.)

Solution. For each of the graphs and tables presented, we want to think of the y-values as *input* and the x-values as *output*. Do determine which ones give x as a function of y, we must decide whether each value of y is paired with exactly one value of x. This is only true for the table in Part (a). In all other tables and graphs, there is at least one value of y that is paired with multiple values of x.

(b) In this table, the value $y = -1$ is paired with $x = -1$ and $x = -4$.

(c) In this graph, the value $y = 6$ is paired with three values of x, namely $x \approx 1.5$, $x = 4.5$, and $x \approx 8.25$. There are many other examples of y-values paired with multiple x-values for this graph.

(d) In this graph, the value $y \approx 4.5$ is paired with an *infinite* number of x-values, namely all values in the interval $9 \leq x \leq 15$.

Parts (e) and (f) give us formulas that relate y and x. In particular, we are given $y = x(8.5 - 2x)(1 - 2x)$ in Part (e) and $y = 2^x$ in Part (f). Does every value of y correspond to only one value of x? The only sure way to determine this when given a formula is to try and solve it for the variable x. If we obtain multiple formulas in the process, then we know that x is *not* a function of y. However, in Part(e), x appears in three factors of a product, so the formula we are dealing with is a *cubic* (third-degree) polynomial. It is very hard to solve cubic polynomials for the variable x, so we will not attempt this approach. In Part (f), the formula $y = 2^x$ is not algebraic at all — there is no purely algebraic way to go about solving for x. (There is no such thing as the xth root of a number.) We will instead graph both functions and check to see if the graphs pass the *horizontal line test*. We imagine horizontal lines moving from bottom to top across the graph. Each horizontal line intersects the y-axis at some value (say k); therefore, the horizontal line is a graphic representation of this particular y-value. Every time a horizontal line $y = k$ crosses the graph of the function, the first coordinate of that intersection point will be an input value for x that produces k as output. Therefore, if horizontal lines never cross the graph more than once, we know that each y-value is paired with (at most) one x-value. (Some horizontal lines might not cross the graph at all.) Of course, this test is of limited value, since we are restricted only to that part of the graph we can see on the calculator. However, it is the best we can do at this time.

(e) The graph of $y = x(8.5 - 2x)(1 - 2x)$ fails the horizontal line test; the line $y = 0$ intersects the graph three times — at the point $(0, 0)$, $(4.25, 0)$, and $(1/2, 0)$. Many other horizontal lines also cross this graph two or more times.

(f) The graph of $y = 2^x$ passes the horizontal line test. This formula therefore defines x as a function of y. (This particular function that gives x in terms of y is called the *base-2 logarithm.*)

(g) In Part (g), weights for bags of concrete are treated as values of x; those weights no heavier than 50 pounds are associated with $y = 1$, while those weights heavier than 50 pounds are associated with $y = 2$. This relationship clearly does not define x as a function of y, because $y = 1$ corresponds to multiple values of x — namely all potential weights between 0 and 50. There is, of course, a caveat to this conclusion — we assume there are bags of concrete available having different weights between 0 and 50 pounds.

(38) [p 114] Evaluate each of the following.

(a) $f(x+2)$ when $f(x) = 4x^2 - 2x + 10$

(b) $h(2x)$ when $h(y) = \dfrac{y^3 - 2y^2 + 4}{2y}$.

Solution. Observe that

$$\begin{aligned} f(x+2) &= 4(x+2)^2 - 2(x+2) + 10 \\ &= 4(x^2 + 4x + 4) - 2(x+2) + 10 \\ &= 4x^2 + 16x + 16 - 2x - 4 + 10 \\ &= 4x^2 + 14x + 22 \end{aligned}$$

$$\begin{aligned} h(2x) &= \frac{(2x)^3 - 2(2x)^2 + 4}{2(2x)} \\ &= \frac{8x^3 - 2(4x^2) + 4}{4x} \\ &= \frac{8x^3 - 8x^2 + 4}{4x} \\ &= \frac{2x^3 - 2x^2 + 1}{x} \end{aligned}$$

(53) [p 120] Use the graphs of f and g provided on Page 120 to answer the following questions.

(a) Approximate the value of the following expressions.

(i) $f(f(3))$
(ii) $g(f(6))$
(iii) $g(g(-2))$
(iv) $g(f(-3))$

(b) Find the value of x that satisfies each of the following equations.

(i) $f(g(x)) = 6$
(ii) $g(f(x)) = 4$

Solution. We begin with Part (a). For part (i), we see from the graphs provided that $f(3) = 6$. Hence, $f(f(3)) = f(6) = 7$. For Part (ii), since we know that $f(6) = 7$, we also know that $g(f(6)) = g(7) = 6$. For Part (iii), we see from the graphs provided that $g(-2) = -1$. Hence, $g(g(-2)) = g(-1) = 1$. For Part (iv), we see from the graphs provided that $f(-3) = -1$. Since we know that $g(-1) = 1$, we may conclude that $g(f(-3)) = 1$.

We conclude with Part (b). For Part (i), we know from the graphs provided that $f(3) = 6$. Thus, we want to solve $g(x) = 3$. Again, the graphs provided tell us that $g(2) = 3$. Hence we may conclude that $x = 2$. For Part (ii), the graphs provided tell us that $g(6) = 4$; hence, we want to solve $f(x) = 6$. However, we already know that $f(3) = 6$; hence, we may conclude that $x = 3$.

(72) [p 124] Determine the process that reverses each of the following functions, then determine if the processes are themselves functions.

(a) $g(x) = 2x + 4$
(b) $h(x) = 2x^3 - 6$
(c) $k(x) = (1/2)x^2 + 12$
(d) $m(x) = 2/(x + 3)$

Solution. The simplest way to determine the reversing process for a function defined by algebra steps is simply to reverse the steps. The best way to reverse the steps is to let y represent the output in the equation defining the function, and then solve for the input variable. (This is not always easy or even possible, but it is the best way to start.) We will adopt this approach for each of these functions.

For Part (a), observe that

$$y = 2x + 4 \Longrightarrow y - 4 = 2x \Longrightarrow \frac{y-4}{2} = x$$

We obtain only one formula as a result of reversing the steps that define the function g. Hence, the reversing process we created defines x as a function of y and therefore serves as the inverse for g.

For Part (b), observe that

$$y = 2x^3 - 6 \Longrightarrow y + 6 = 2x^3 \Longrightarrow \frac{y+6}{2} = x^3 \Longrightarrow \sqrt[3]{\frac{y+6}{2}} = x$$

We obtain only one formula as a result of reversing the steps that define the function h. Hence, the reversing process we created defines x as a function of y and therefore serves as the inverse for h.

For Part (c), observe that

$$y = \left(\frac{1}{2}\right) x^2 + 12 \Longrightarrow y - 12 = \left(\frac{1}{2}\right) x^2 \Longrightarrow 2(y - 12) = x^2 \Longrightarrow \pm\sqrt{2(y-12)} = x$$

In this case, we obtain two formulas for x as a result of reversing the steps that define the function k. There are no restrictions on the domain of k given in the problem, so there is no way to eliminate one of these possible formulas. Hence, we must conclude that the reversing process does not define x as a function of y. The function k does not have an inverse.

For Part (d), observe that

$$y = \frac{2}{x+3} \Longrightarrow y(x + 3) = 2 \Longrightarrow x + 3 = \frac{2}{y} \Longrightarrow x = \frac{2}{y} - 3$$

We obtain only one formula as a result of reversing the steps that define the function m. Hence, the reversing process we created defines x as a function of y and therefore serves as the inverse for m.

We can now use standard inverse notation to describe the inverses of the functions g, h, and m. For example, the inverse of the function g can be written

$$g^{-1}(y) = \frac{y-4}{2}$$

When no real-world meaning as been assigned to the input and output variables for a function, it is a common (but not very laudable) practice to use the same input variable for the inverse function as for the function itself. Adhering to this custom, we could write

$$g^{-1}(x) = \frac{x-4}{2} \qquad h^{-1}(x) = \sqrt[3]{\frac{x+6}{2}} \qquad m^{-1}(x) = \frac{2}{x} - 3$$