INTRODUCING FUNCTIONS

THE DEFINITION

When two quantities vary together, there is a rule or process that takes you *from* the values of one quantity *to* the values of the other quantity.

This process may be summarized verbally, by a graph, by a table of data, or by a formula.

The quantity values the process starts with are called ***inputs*** into the process.

The quantity values the process ends with are called ***outputs*** from the process.

A process is called a **function** when every *input* value into the process produces exactly one *output* value from the process.

When we are discussing a process relating two varying quantities, it is important to specify the input and output values.

Let’s consider the relationship presented by the following table, which shows the temperature in degrees Fahrenheit for various times in Murfreesboro on July 12, 2012.

Temperature (Degrees F)	88	90	93	94	96	94	90	87
Hours since Noon	-2	-1	0	1	3	4	5	7

This table summarizes *two* processes.

Process 1 The process that *starts* with the temperature in degrees Fahrenheit and *ends* with the number of hours since noon.

Process 2 The process that *starts* with the number of hours since noon and *ends* with temperature in degrees Fahrenheit

We say that Process 1 gives the number of hours since noon *in terms of* the temperature in degrees Fahrenheit.

We say that Process 2 gives the temperature in degrees Fahrenheit *in terms of* the number of hours since noon.

When verbally describing a process relating two changing quantities, the phrase "**in terms of**" comes before the **input quantity**.

The phrase "**with respect to**" is also used to point out the input quantity.

Process 1 is NOT a function, since there are values of the input quantity that produce multiple values of the output quantity.

- For example, the input value 90 F produces two output values, namely -1 and 6 hours since noon.

Process 2 IS a function, since every value of the input quantity produces only one value of the output quantity.

We would say that Process 2 gives the temperature in degrees Fahrenheit ***as a function of*** the number of hours since noon.

It is common practice to use letters as names for processes that are functions.

For example, we could let F represent Process 2 that gives the temperature in degrees Fahrenheit with respect to the number of hours since noon.

In this context, **the letter *F* is not a variable**. It is the name of the process.

We can assign variables to the two quantities related by Process 2.

- Let h represent the number of hours since noon.
- Let t represent the temperature in degrees Fahrenheit

The relationship between the quantities h and t is expressed by the formula

$$t = F(h)$$

This formula $t = F(h)$ is shorthand for the following sentences:

- The variable t is a function of the variable h through the process named F.
- Under the process named F, an input value of h produces an output value of t.

The formula $t = F(h)$ is read "***t* is equal to *F* of *h***."

The letter *F* is the name of a process … however, the expression *F*(*h*) is the name of a variable. It is another name for the output variable *t*.

The expression $F(h)$ cannot be broken up. It is one symbol, and the pieces cannot be separated.

The expression $F(h)$ is not a product. Remember, F is not a variable.

The formula $\boldsymbol{t = F(h)}$ is called *function notation*, and it can be used to represent many relationships in the table.

VERBAL	FUNCTION NOTATION
• "One hour before noon, the temperature was 90F."	$90 = F(-1)$
• "When was the temperature 94F?"	Solve $94 = F(h)$
• "The change in temperature as the number of hours since noon increases from $h = -2$ to $h = 3$"	$\Delta t = F(3) - F(-2)$

Important things to notice in the last example above:

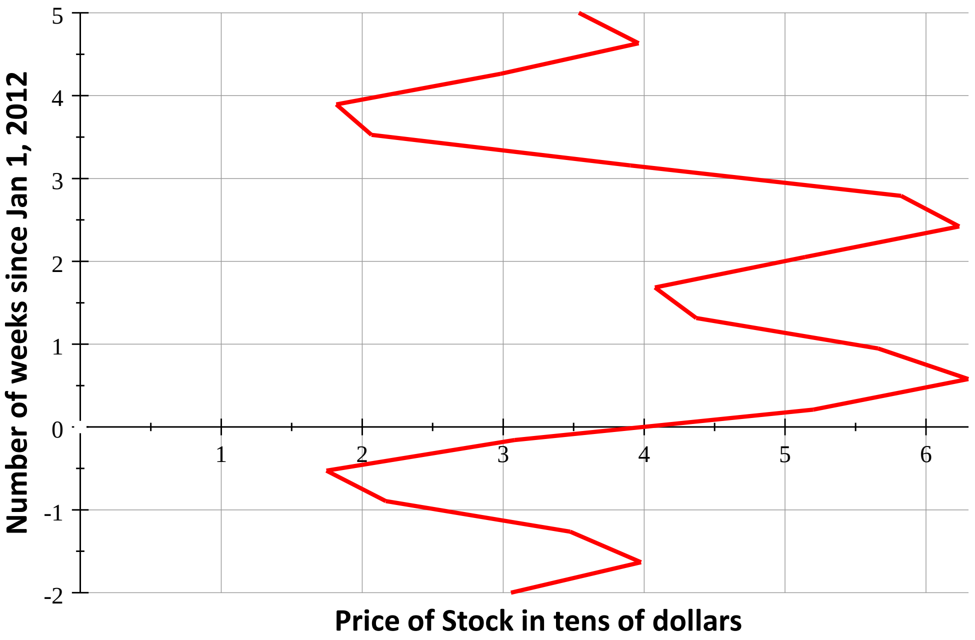
1. Looking at the table, notice that $F(3) - F(-2)$ is NOT equal to $F(3 - (-2))$

$$F(3) - F(-2) = 96 - 88 = 8 \qquad\qquad F(3 - (-2)) = F(5) = 90$$

2. Since F is not a variable, it is NOT correct to write $\Delta F = F(3) - F(-2)$. It would be correct to write $\Delta F(h) = F(3) - F(-2)$, since $t = F(h)$.

The graph on the right shows the relationship between the price of a certain stock and the number of weeks since January 1, 2012.

Let *s* represent the price of the stock in tens of dollars, and let *w* represent the number of weeks since January 1, 2012.



This graph summarizes two processes.

Process 1 gives the values of *t* in terms of the values of *w*. In this process, *w* is the input variable, and *t* is the output variable.

Process 2 gives the values of *w* in terms of the values of *t*. In this process, *t* is the input variable, and *w* is the output variable.

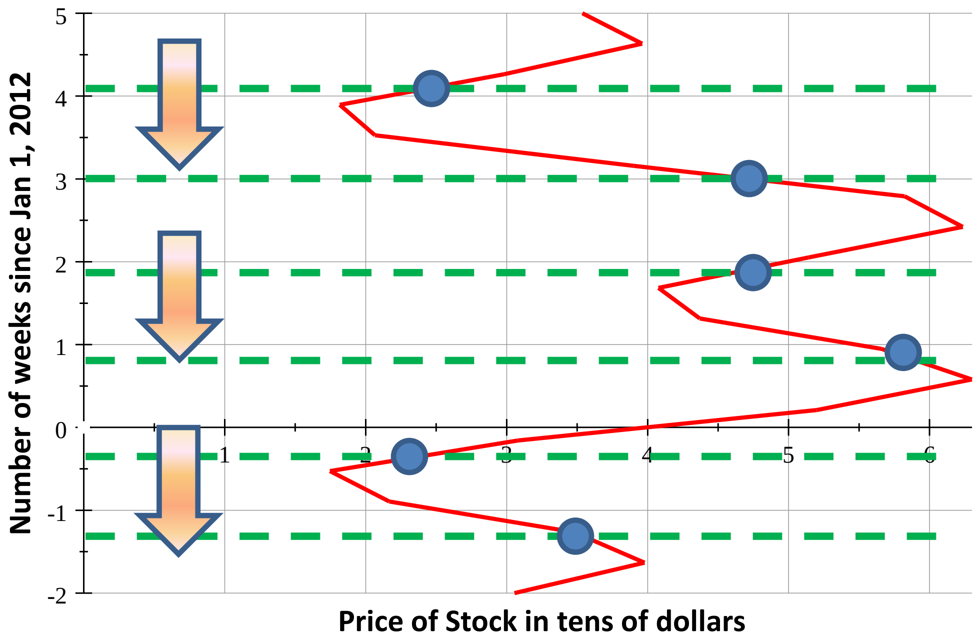
Does one of these processes represent a function?

- When working with a graph, imagine lines drawn ***perpendicular to the input axis***.
- If any ***one of these lines intersects the graph more than once***, then the process is not a function.

Consider Process 1. In this process, the vertical axis is the input axis since the values of *w* lie on this axis.

Lines perpendicular to this axis will be horizontal.

A horizontal line sliding down diagram from top to bottom will never cross the graph more than once.



Horizontal lines represent input values for Process 1.

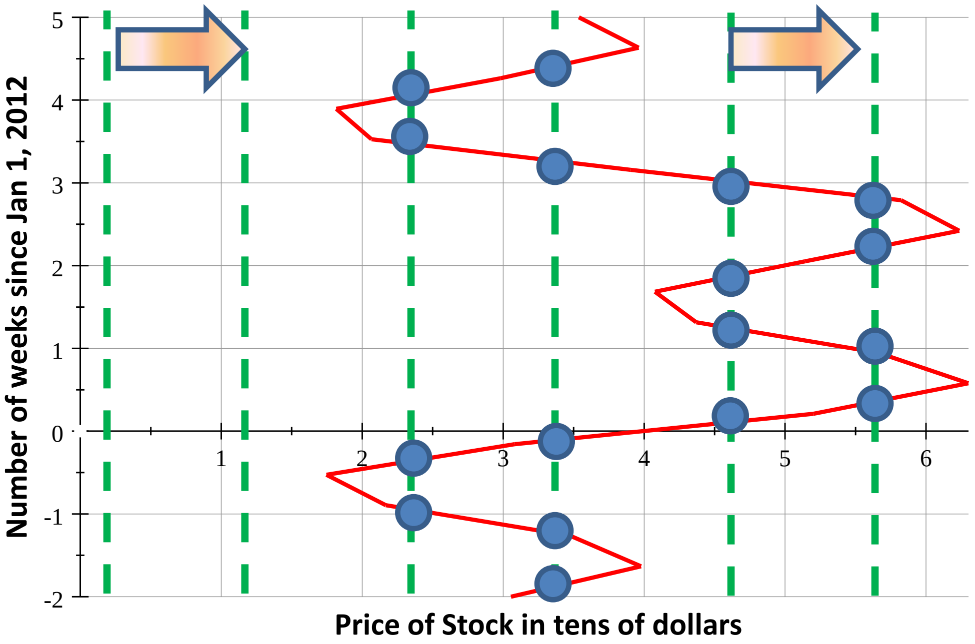
Every time a horizontal line crosses the graph, Process 1 pairs that input value with an output value.

Every input value is paired with only one output value, so Process 1 represents ***t*** *as a function of* ***w***.

Consider Process 2. In this process, the horizontal axis is the input axis since the values of *t* lie on this axis.

Lines perpendicular to this axis will be vertical.

As we move left-to-right across diagram *some* vertical lines will cross the graph more than once.



Vertical lines represent input values for Process 2.

Every time a vertical line crosses the graph, Process 2 pairs that input value with an output value.

Some input values are paired with more than one output value, so Process 2 **DOES NOT** represent ***w*** *as a function of* ***t***.

In the table and graph examples, the processes relating the changing quantities are not explicitly stated.

We don't know the actual *rule* behind these processes, only the outcome of the rule.

If the outcome of a process is an algebraic formula then the actual rule behind the process can be found.

Let *x* and *y* represent the values of two varying quantities, and suppose these quantities are related by the formula

$$y = \frac{3 - 2x}{\sqrt{x - 1}}$$

In this formula, the values of *x* serve as inputs into the process, and the values of *y* serve as outputs from the process.

We know this because the values of *y* are determined by our choice of value for *x*.

The algebra steps required to transform the values of *x* into the values of *y* constitute the rule behind this process.

There are two parallel sets of steps that are combined in the final step to produce the output value of *y*.

Working in the numerator

STEP 1a Start with the value of *x* and multiply it by 2.

STEP 1b Take the result of Step 1b and subtract it from 3.

Working in the denominator

STEP 2a Start with the value of *x* and subtract 1 from it.

STEP 2b Take the result of Step 2a and take the square root of it.

STEP 3 Take the result of Step 1b and divide it by the result of Step 2b

These steps make up the actual rule behind this process.

When we start with a value of *x*, the process can produce only one value of *y*, so this process gives *y* *as a function of* *x*.

The process relating the values of *x* and *y* summarized by the formula

$$y = \frac{3 - 2x}{\sqrt{x - 1}}$$

gives *y* as a function of *x*. We can therefore give a name to this process. Let's call this process ***f***.

The letter ***f*** is the name we have given to the steps that take us from the value of the input *x* to the value of the output *y*.

Letting ***f*** represent this process automatically gives us another name for the output variable.

$$y = f(x) = \frac{3 - 2x}{\sqrt{x - 1}}$$

- What is the meaning of the expression $-\frac{17}{3} = f(10)$?

This expression means that an input value of *x* = 10 into the process named *f* produces an output value of $y = -\frac{17}{3}$.

- Use the definition of the function f to evaluate $f(5)$ and $f(-1)$.

The function f is defined to be the process that gives the values of y as a function of the values of x summarized by the formula

$$y = f(x) = \frac{3-2x}{\sqrt{x-1}}$$

$$f(5) = \frac{3-2(5)}{\sqrt{5-1}} \Rightarrow f(5) = -\frac{7}{2}$$

$$f(-1) = \frac{3-2(-1)}{\sqrt{-1-1}} \Rightarrow f(-1) = -\frac{9}{\sqrt{-2}}$$ UNDEFINED

The output y = -7/2 is produced by the input x = 5, but there is ***no output*** produced by x = -1.

The **DOMAIN** of a function is the set of all values of the input variable that produce meaningful output in the process that defines the function.

The value x = 5 is part of the domain of f, while the value x = -1 is not.

If a function represents the process behind an algebraic formula, then its domain will be the set of all real numbers that

- do not result in division by 0 **OR**
- do not result in the square root of a negative number

when used as input values in the function.

- What is the domain of the function behind the formula $y = f(x) = \dfrac{3-2x}{\sqrt{x-1}}$?

Since $\sqrt{0} = 0$, we know that division by 0 will occur when

$$x - 1 = 0$$

This means $x = 1$ is not part of the domain of the function f.

The square root of a negative number will occur when

$$x - 1 < 0$$

This means all values of x less than 1 are also not part of the domain of f.

The domain of f will be the set of all values $x > 1$.

- What is the domain of the function behind the formula $u = g(t) = \frac{\sqrt{15+2t}}{t^2-25}$?

First, notice that g is the name of the function behind the formula.

The process named g gives the variable u as a function of the variable t.

The domain for this function will be a set of values for the variable t.

Division by 0 will occur when $t^2 - 25 = 0$, and this happens when $t = \pm 5$.
Therefore, t = 5 and t = -5 are not part of the domain of g.

The square root of a negative number will occur when $15 + 2t < 0$, and this happens when $t < -7.5$. Therefore, no value of t smaller than -7.5 is part of the domain of g.

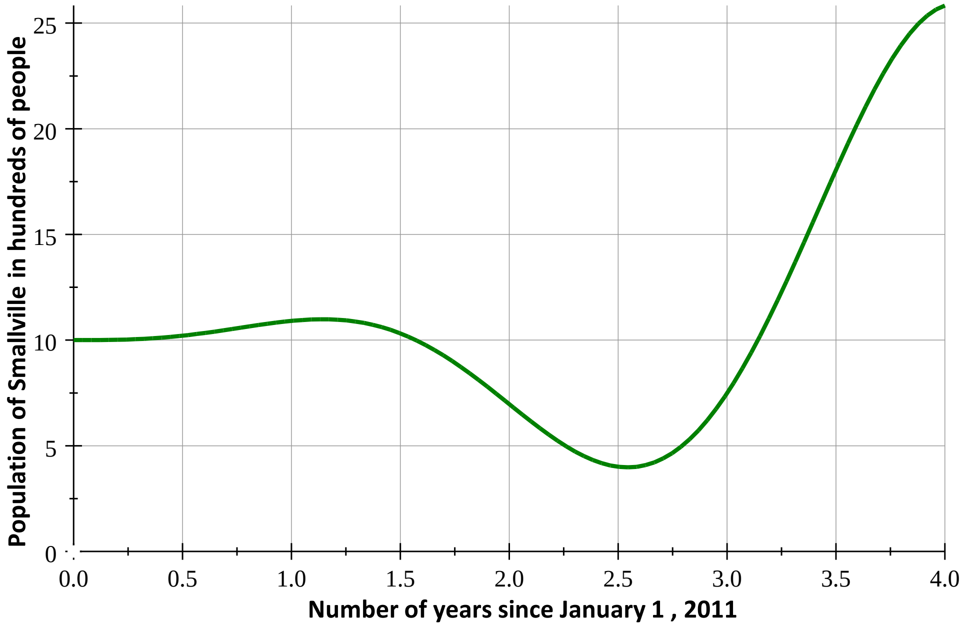
The domain of the function g will be all values $t \geq -7.5$ ***except*** for t = 5 and t = -5.

WORKING WITH FUNCTION NOTATION

When a process represents a function and we want to graph the outcomes of the process, we usually use the horizontal axis for the input variable and the vertical axis for the output variable.

The graph on the right shows the population of Smallville in terms of the number of years passed since January 1, 2011

Let *P* represent the population of Smallville in hundreds of people, and let *y* represent the number of years passed since January 1, 2011.



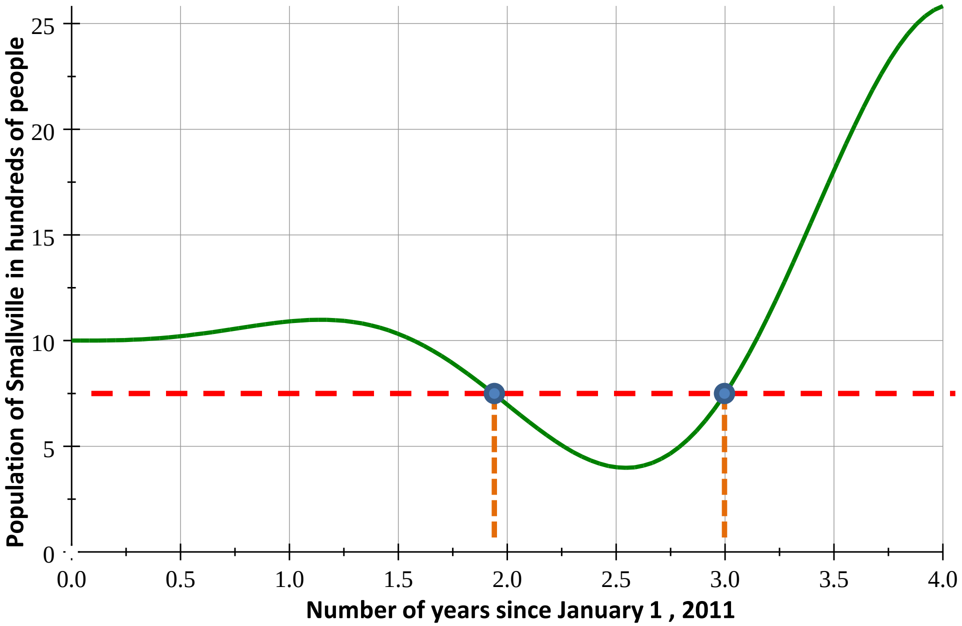
Let *h* represent the function that gives the values of *P* in terms of the values of *y*.

- In this context, what does it mean to solve the equation $7.5 = h(y)$?

First, note that there is only one unknown in this problem. The letter *h* is the name of the process that transforms the values of *y* into the values of *P*. Therefore, *h* is NOT an unknown --- only *y* is an unknown.

The equation $7.5 = h(y)$ means "The value 7,500 people is the output of the function *h* when the input value is *y*." Therefore, solving this equation means finding the number of years after January 1, 2011 when the population is 7,500 people.

To solve the equation, draw a horizontal line from *P* = 7.5 and look for points where the line crosses the graph.



The crossing points tell us that $7.5 = h(y)$ when $y \approx 1.9$ and $y \approx 3.0$.

It would **NOT** be correct to say that $y \approx 2012$ and $y \approx 2014$ because *y* is defined to be *the number of years* since January 1, 2011 … NOT the year.

- Use function notation to represent the change in the value of *P* as the value of *y* increases from *y* = 1.5 to *y* = 3.5 years since January 1, 2011.

When *y* = 1.5 we know that $P = h(1.5)$. When *y* = 3.5 we know that $P = h(3.5)$.

The change in *P* is therefore $\Delta P = h(3.5) - h(1.5)$

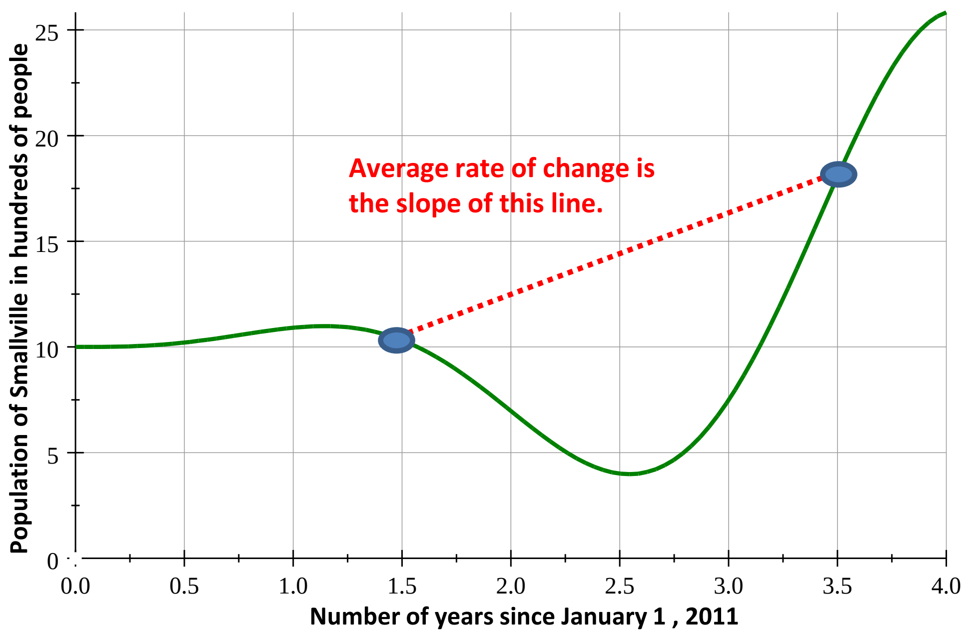
- Use function notation to express the average rate of change for the function *h* as the value of *y* increases from *y* = 1.5 to *y* = 3.5 years since January 1, 2011.

The average rate of change will be $\frac{\Delta P}{\Delta h} = \frac{h(3.5)-h(1.5)}{3.5-1.5}$

- What is the meaning of average rate of change in this context?

Average rate of change is the slope of the line segment connecting the points (1.5 , *h*(1.5)) and (3.5 , *h*(3.5)) .

Average rate of change is the constant rate the population would have to change at in order to go from *h*(1.5) to *h*(3.5) in two years.



- What is the meaning of the expression $h(y+3)$?

We know that *y* represents the number of years passed since January 1, 2011.

The expression $y+3$ means "three years after *y* years has passed since January 1, 2011."

The expression $h(y+3)$ therefore means "the population of Smallville in hundreds three years after *y* years has passed since January 1, 2011."

- What is the meaning of the expression $h(y)+3$?

The expression *h*(*y*) means "the population of Smallville in hundreds *y* years after January 1, 2011."

Therefore, the expression $h(y)+3$ means "three more people than the population of Smallville in hundreds *y* years after January 1, 2011."