L'Hôpital's Rule

In Chapter 2 we encountered an important indeterminate form when working with limits of algebraic functions:

- We say that $\lim_{x\to a} \frac{f(x)}{g(x)}$ has indeterminate form $0/0$ provided $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = 0$.

This type of limit has played a key role in differentiation — indeed, the computation of every derivative is ultimately based upon a limit of this form. When we worked with these limits, we usually employed the "factor and cancel" approach for evaluating them. To evaluate a limit of the form $0/0$, we had to factor the numerator and denominator and cancel terms. For example,

$$\lim_{x\to 1} \frac{x^2+x-2}{x^2+2x-3} = \lim_{x\to 1} \frac{(x-1)(x+2)}{(x-1)(x+3)} = \lim_{x\to 1} \frac{x+2}{x+3} = \frac{3}{4}$$

It turns out that there are many other indeterminate forms that limits can take. We seldom encounter most of these in practice, but there is one that frequently appears in applications:

- We say that $\lim_{x\to\infty} \frac{f(x)}{g(x)}$ has indeterminate form ∞/∞ if $\lim_{x\to\infty} f(x) = \pm\infty$ and $\lim_{x\to\infty} g(x) = \pm\infty$.

When a limit of the form ∞/∞ involves an algebraic function, we can evaluate it easily by simply ignoring all but the highest powers of the variable in the numerator and the denominator. For example,

$$\lim_{x\to+\infty} \frac{x^2+x-2}{x^2+2x-3} = \lim_{x\to+\infty} \frac{x^2}{x^2} = 1$$

The reason this trick works is quite simple: As x grows larger and larger in magnitude in an algebraic function, the highest powers of the variable eventually come to dominate all of the other powers. Hence, as x grows, we can ignore the contribution from the lower powers. It is very important to remember that this trick ONLY works for limits of algebraic functions having indeterminate form ∞/∞.

It turns out that there is a general rule which gives us a single method for handling both types of indeterminate forms — not only for algebraic functions, but for ratios of any kind of functions. This rule is known as L'Hôpital's Rule. It is named for a prominent French mathematician who publicized but did not invent the method.

Theorem Let f and g be differentiable functions and let a be a real number or possibly $\pm\infty$. If it is the case that $\lim_{x\to a}(f(x)/g(x))$ has indeterminate form $0/0$ or ∞/∞, then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

Notice that applying L'Hôpital's Rule is not the same as differentiating the quotient f/g. When applying L'Hôptial's Rule, you must differentiate the numerator and denominator *separately.*

Example 1. Evaluate $\lim_{x\to 1} \frac{x^2+x-2}{x^2+2x-3}$.

Solution. This limit is an indeterminate form of type $0/0$. Hence, we know that

$$\lim_{x\to 1} \frac{x^2+x-2}{x^2+2x-3} \underset{\text{LHR}}{=} \lim_{x\to 1} \frac{2x+1}{2x+2} = \frac{3}{4}$$

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Notice how much faster applying L'Hôptial's Rule can be (opposed to factoring). It can be a major help when it is difficult to factor the one of the functions.

Problem 1. Evaluate $\lim_{x\to 2} \dfrac{x^4 - 2x^3 - x^2 + 2x}{x^4 - 16}$.

L'Hôpital's Rule extends to any indeterminate limits. We can use this method to work with transcendental functions as well as algebraic functions.

Example 2. Evaluate $\lim_{x\to 0} \dfrac{\ln(1+x)}{\sin(x)}$.
Solution. This limit has indeterminate form 0/0. Hence, we know that

$$\lim_{x\to 0} \frac{\ln(1+x)}{\sin(x)} \underset{\text{LHR}}{=} \lim_{x\to 0} \frac{1/(1+x)}{\cos(x)} = \lim_{x\to 0} \frac{1}{(x+1)\cos(x)} = 1$$

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Problem 2. Evaluate $\lim_{x\to 0^+} \dfrac{\ln(3x)}{\ln(2x)}$.

Example 3. Evaluate $\lim_{x\to 1} \dfrac{\ln(x)}{1+\cos(\pi x)}$.

This limit is an indeterminate form of type 0/0. Hence, we know that

$$\lim_{x\to 1^+} \frac{\ln(x)}{1+\cos(\pi x)} \underset{\text{LHR}}{=} \lim_{x\to 1^+} \frac{1/x}{-\pi\sin(\pi x)} = \lim_{x\to 1^+} \left(-\frac{1}{\pi x\sin(\pi x)}\right)$$

Now, this limit is no longer indeterminate. The denominator approaches 0 while the numerator remains fixed at 1. Hence, this limit is infinite. In fact, as x approaches 1 *from the right*, the expression $x\sin(\pi x)$ is negative. Hence, we know that

$$\lim_{x\to 1^+} \frac{\ln(x)}{1+\cos(\pi x)} = \lim_{x\to 1^+} \left(-\frac{1}{\pi x\sin(\pi x)}\right) = +\infty$$

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L'Hôpital's Rule may be applied repeatedly to a limit, as long as you have an indeterminate form.

Example 4. Evaluate $\lim_{x\to +\infty} \dfrac{(\ln(x))^2}{x}$.

Solution. This is an indeterminate form of type ∞/∞. Hence, we know that

$$\lim_{x\to +\infty} \frac{(\ln(x))^2}{x} \underset{\text{LHR}}{=} \lim_{x\to +\infty} \frac{2(\ln(x)/x)}{1} = \lim_{x\to +\infty} 2\frac{\ln(x)}{x}$$

The last limit is still an indeterminate form (of type ∞/∞), hence we must apply L'Hôptial's Rule a second time. Observe that

$$\lim_{x\to+\infty} 2\frac{\ln(x)}{x} \underset{\text{LHR}}{=} \lim_{x\to+\infty} \frac{2/x}{1} = \lim_{x\to+\infty} \frac{2}{x} = 0$$

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Problem 3. Evaluate $\lim_{x\to 0} \frac{\sin(x)-x}{x^2}$.

It is important to remember that L'Hôpital's Rule can *only* be applied to indeterminate forms of type $0/0$ or ∞/∞. For example, the limit

$$\lim_{x\to 0} \frac{x+\sin(x)}{x+\cos(x)}$$

is not an indeterminate form, since $\lim_{x\to 0}(x+\cos(x)) = 1$. In cases like this, we simply fall back on the techniques of Chapter 2. In particular, we just evaluate the limit directly by plugging in $x = 0$:

$$\lim_{x\to 0} \frac{x+\sin(x)}{x+\cos(x)} = \frac{0}{1} = 0$$

HOMEWORK: Section 4.4, Page 311 Problems 9, 10, 15, 16, 17, 18, 19, 21, 22, 23