COMPUTING WITH LIMITS

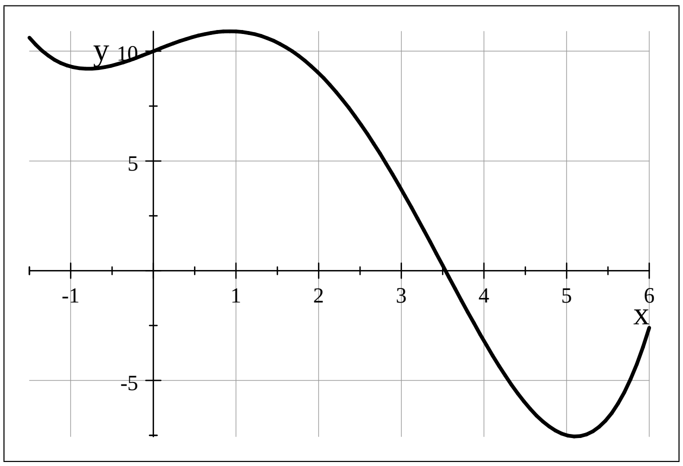
Suppose that f represents a function of the input variable x. We say that f is *continuous* at the value $x = a$ provided the following conditions are met:

- The function f is defined at $x = a$; that is, $f(a)$ exists.
- There are no jumps in the graph of f at $x = a$.
- There are no tears (vertical asymptotes) in the graph of f at $x = a$.

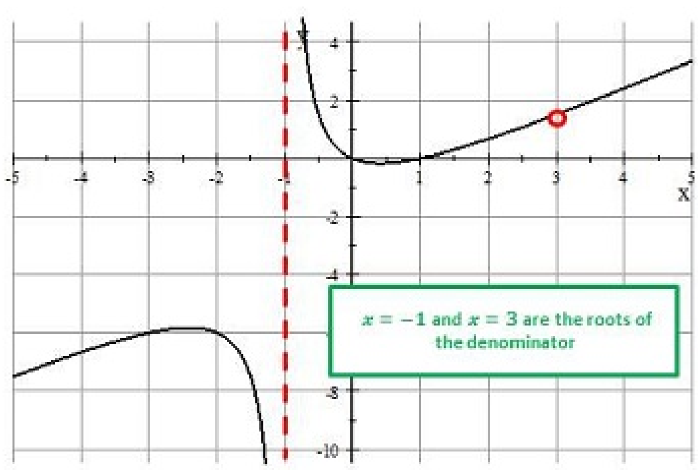
If any one of these conditions fails, then we say that f has a *discontinuity* at $x = a$.

If a function f is continuous at $x = a$, then you can draw the graph of f through the point $(a, f(a))$ without taking your pencil off the paper. With this visualization in hand, it is easy to see that

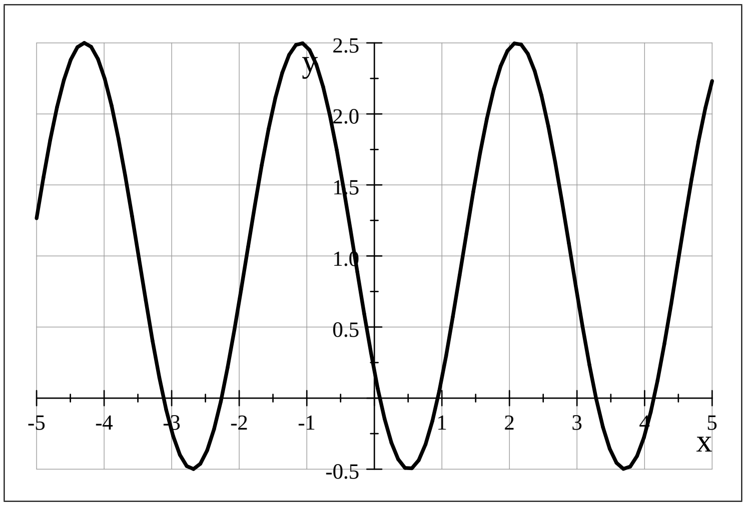
(A) Polynomial functions are continuous at any input value. For example, here is the graph of the polynomial $y = f(x) = \frac{1}{10}\left(x^4 - 7x^3 + 15x\right) + 10$ on the intervval $-1 \leq x \leq 6$.



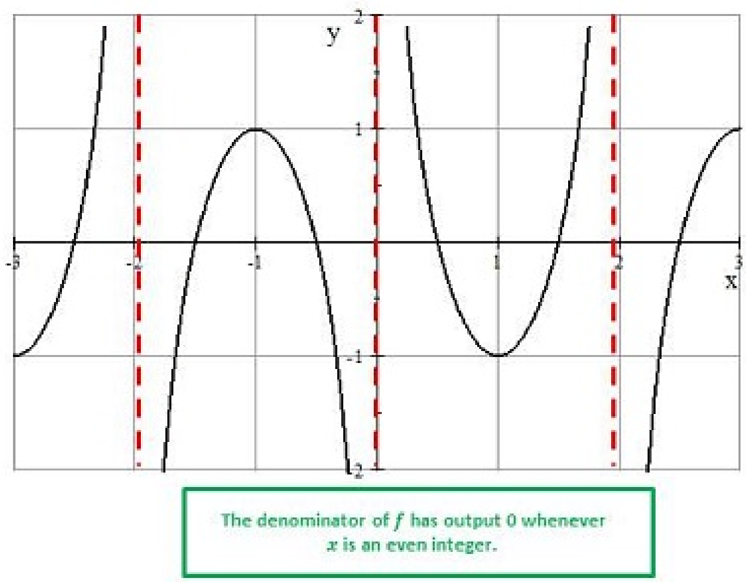
(B) A ratio of two polynomials (a rational function) is continuous at any input value that is *not* a root of the denominator. For example, here is the graph of the rational function $y = f(x) = \dfrac{x^4 - 3x^3 - x^2 + 3x}{x^3 - x^2 - 5x - 3}$ on the interval $-5 \leq x \leq 5$.



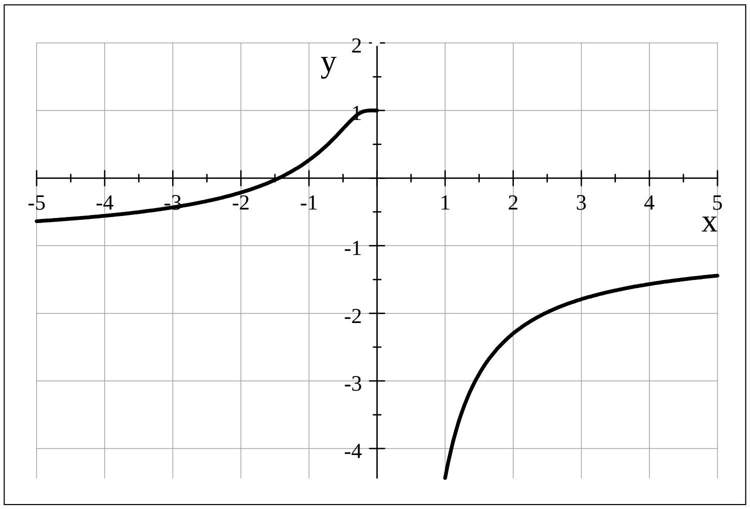
(C) As long as its input is a continuous function, sinusoid functions are continuous at any input value. For example, consider the sinusoid $y = f(x) = 1 - \frac{3}{2}\cos\left(\frac{5\pi}{8}x - 1\right)$. The input into the sinusoid is the polynomial $p(x) = \frac{5\pi}{8}x - 1$. Since p is continuous for every input value of x, the sinusoid f will also be continuous for every input value of x. Here is the graph of the sinusoid on the interval $-5 \leq x \leq 5$.



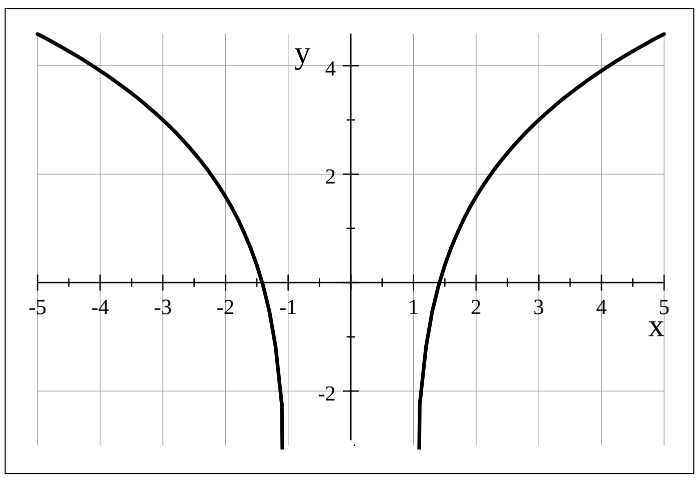
(D) Ratios of continuous sinusoid functions will be continuous at any input value that does not produce an output of 0 from the function in the denominator. For example, here is the graph of $y = f(x) = \dfrac{\cos(\pi x)}{\sin(\pi x/2)}$ on the interval $-3 \leq x \leq 3$.



(E) As long as its input is a continuous function, an exponential function is continuous at any input value. For example, consider the exponential function $y = f(x) = 1 - 2e^{1/x}$. The input into this exponential function is the rational function $r(x) = \frac{1}{x}$. Since r is continuous for every input value of x EXCEPT for $x = 0$, the exponential function f will also be continuous for every input value of x EXCEPT for $x = 0$. Here is the graph of the exponential function f on the interval $-5 \leq x \leq 5$.



(F) As long as its input is a continuous function with *positive output*, a logarithmic function is continuous at any input value. For example, consider the logarithmic function $y = f(x) = \log_2(x^2 - 1)$. The input into this exponential function is the polynomial $p(x) = x^2 - 1$. Since p is continuous for all input values of x, the logarithmic function f will be continuous as long as the output from p is positive. Since p has positive output as long as $x < -1$ or $x > 1$, the logarithmic function f will also be continuous on these rays. Here is the graph of the exponential function f on the interval $-5 \leq x \leq 5$.



Problem 1. Does the following piecewise-defined function have any discontinuities? If so, at what input values do these discontinuities occur?

$$f(x) = \begin{cases} 2x - 1 & \text{if} \quad x < 1 \\ x & \text{if} \quad 1 \leq x \leq 3 \\ x^2 - 5 & \text{if} \quad 3 < x \end{cases}$$

Problem 2. Does the following piecewise-defined function have any discontinuities? If so, at what input values do these discontinuities occur?

$$g(t) = \begin{cases} \frac{1}{t+2} & \text{if} \quad t \leq -1 \\ \sin(\pi t) & \text{if} \quad -1 < t \leq 0 \\ \ln(t) & \text{if} \quad 0 < t \end{cases}$$

HOMEWORK: Section 2.5 page 124, Problems 17, 18, 19, 20, and 21. (Look for jumps, tears, or holes in the graphs of these functions.)

DIRECT SUBSTITUTION PRINCIPLE

If a function f is continuous at $x = a$, then f has output at $x = a$, and the graph of f "wants to pass through" thepoint $(a, f(a))$ — there can be no holes, jumps, or tears in the graph at the point $(a, f(a))$. In other words, if f is continuous at $x = a$, then

$$\lim_{x \longrightarrow a} f(x) = f(a)$$

Problem 3. Compute $\displaystyle\lim_{x \longrightarrow 3} \frac{2x^3 - 2x + 5}{x^2 - 1}$.

Problem 4. Compute $\displaystyle\lim_{t \longrightarrow \pi/4} \tan(t)$.

Problem 5. Compute $\displaystyle\lim_{y \longrightarrow -1} 5^{-y}$.

Limits are"blind" to the output of a function at a single input value. That is, the limiting process for a function f as the input variable x approaches a specific value a does not care what the actual output of f is at $x = a$ (or even it if exists). The limiting process only "feels" the behavior of f as the values of x get very close to a. This tells us that

- If two functions f and g have the same output at every input value *except* at $x = a$, then the limits of these functions as x approaches a will be the same; that is,

$$\lim_{x \longrightarrow a} g(x) = \lim_{x \longrightarrow a} f(x)$$

Example 1 *Compute* $\displaystyle\lim_{x \longrightarrow 4} \frac{x^2 - 16}{x^2 - 7x + 12}$.

Solution. In this case, the function

$$f(x) = \frac{x^2 - 16}{x^2 - 7x + 12}$$

has a discontinuity at $x = 4$, because this value of x is a root of the denominator (so that $f(4)$ is undefined). Because of this, we cannot apply the Direct Substitution Principle. However, notice that

$$\frac{x^2 - 16}{x^2 - 7x + 12} = \frac{(x-4)(x+4)}{(x-4)(x-3)} = \frac{x+4}{x-3} \quad \text{so long as } x \neq 4$$

Therefore, if we let

$$g(x) = \frac{x+4}{x-3}$$

we know that f and g have the same output at every input value *except* the input value $x = 4$. Consequently, the limiting process cannot tell the difference between these functions at $x = 4$. Furthermore, the function g is continuous at $x = 4$; so we know the Direct Substitution Principle *applies to the function* g.

$$\lim_{x \longrightarrow 4} \frac{x^2 - 16}{x^2 - 7x + 12} = \lim_{x \longrightarrow 4} f(x) = \lim_{x \longrightarrow 4} g(x) = \lim_{x \longrightarrow 4} \frac{x+4}{x-3} = \frac{4+4}{4-3} = 8$$

****

The function f in the previous example has a discontinuity at $x = 4$. However, by simplifying the formula for f, we could *remove* the cause of the discontinuity at $x = 4$ (namely the division by 0 caused by the factor $x - 4$ in the denominator). The simplified formula represents a new function g that is continuous at $x = 4$. For this reason, we say that f has a *removable discontinuity* at $x = 4$.

Problem 6. Compute $\lim_{x \longrightarrow 2} \dfrac{3x - 6}{x^2 + 2x - 8}$.

Problem 7. Compute $\lim_{z \longrightarrow -1} \dfrac{z^3 + z}{z^2 - 1}$.

HOMEWORK: Section 2.3, pp. 102-103 Problems 11, 12, 15, 20, 23, and 24. (Rewrite Problems 23 and 24 as a single fraction first.)

Example 2 *The function* $f(x) = \dfrac{x^2 - 16}{x^2 - 7x + 12}$ *has a discontinuity at* $x = 3$*. Is this a removable discontinuity?*

Solution. First, the reason that f has a discontinuity at $x = 3$ is because this input value results in division by 0. Therefore, the function f is undefined when $x = 3$ and cannot be continuous at this input value. Is it possible to simplify the output formula for f so that we can eliminate this division by 0 at $x = 3$? We already know that

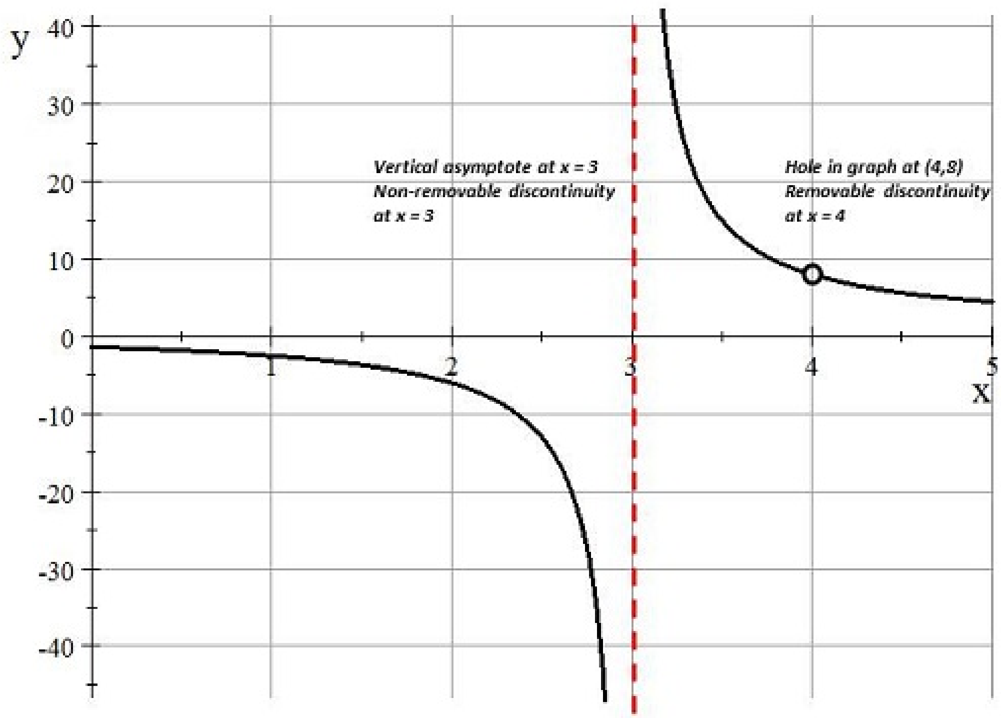
$$\frac{x^2 - 16}{x^2 - 7x + 12} = \frac{(x-4)(x+4)}{(x-4)(x-3)} = \frac{x+4}{x-3} \quad \text{so long as } x \neq 4$$

There is no further simplification we can perform on the output formula for f, and the simplified formula is still undefined at the input value $x = 3$. Therefore, f has a *non-removable* discontinuity at $x = 3$.

****

If you use your graphing calculator to sketch the graph of f in the previous example, you will see that f has a vertical asymptote (a tear in the graph) at $x = 3$. This is usually, but not always, the case for non-removable discontinuities.

If we apply the limiting process to the function f by letting the input values for x approach 3, what happens? Here is the graph of the function f near its discontinuities.



Graph of f

Based on the graph, none of the following limits exist:

$$\lim_{x \longrightarrow 3} f(x) \qquad \lim_{x \longrightarrow 3^-} f(x) \qquad \lim_{x \longrightarrow 3^+} f(x)$$

These limits fail to exist, because, in each case, as we allow the input values for x to approach 3 from the left or from the right, the output of f does not approach a fixed value. (The outputs keep getting larger and larger in magnitude.) However, there is a consistent *behavior* to the outputs of the function f as we allow the input values for x to approach 3 from the left or from the right.

- As we allow the input values for x to approach 3 from the right, the outputs of f become more and more positive. For this reason, it is customary to write

$$\lim_{x \longrightarrow 3^+} f(x) = +\infty$$

- As we allow the input values for x to approach 3 from the left, the outputs of f become more and more negative. For this reason, it is customary to write

$$\lim_{x \longrightarrow 3^-} f(x) = -\infty$$

There is a troubling inconsistency to this custom, unfortunately. The symbols $\pm\infty$ DO NOT REPRESENT NUMBERS. These symbols merely indicate that there is consistent behavior to the outputs of f in the limiting process.

Problem 8. Consider the function $f(x) = \dfrac{x^3 - 2x}{x^2 - x - 2}$.

Part (a): Does f have any removable discontinuities? If so, what is the limit of the outputs of f when we allow x to approach these values?

Part (b): Does f have any non-removable discontinuities? If so, what is the limit of the outputs of f when we allow x to approach these values from the left or from the right?

Problem 9. Consider the function $g(t) = \dfrac{t^2 - t - 6}{t - 3}$.

Part (a): Does g have any removable discontinuities? If so, what is the limit of the outputs of g when we allow t to approach these values?

Part (b): Does g have any non-removable discontinuities? If so, what is the limit of the outputs of g when we allow t to approach these values from the left or from the right?

Problem 10. Consider the function $h(a) = a - \dfrac{2}{1 - a}$.

Part (a): Does h have any removable discontinuities? If so, what is the limit of the outputs of h when we allow a to approach these values?

Part (b): Does h have any non-removable discontinuities? If so, what is the limit of the outputs of h when we allow a to approach these values from the left or from the right?

Problem 11. Consider the function $f(x) = \dfrac{x^3 - 4x}{x^3 + 2x^2}$.

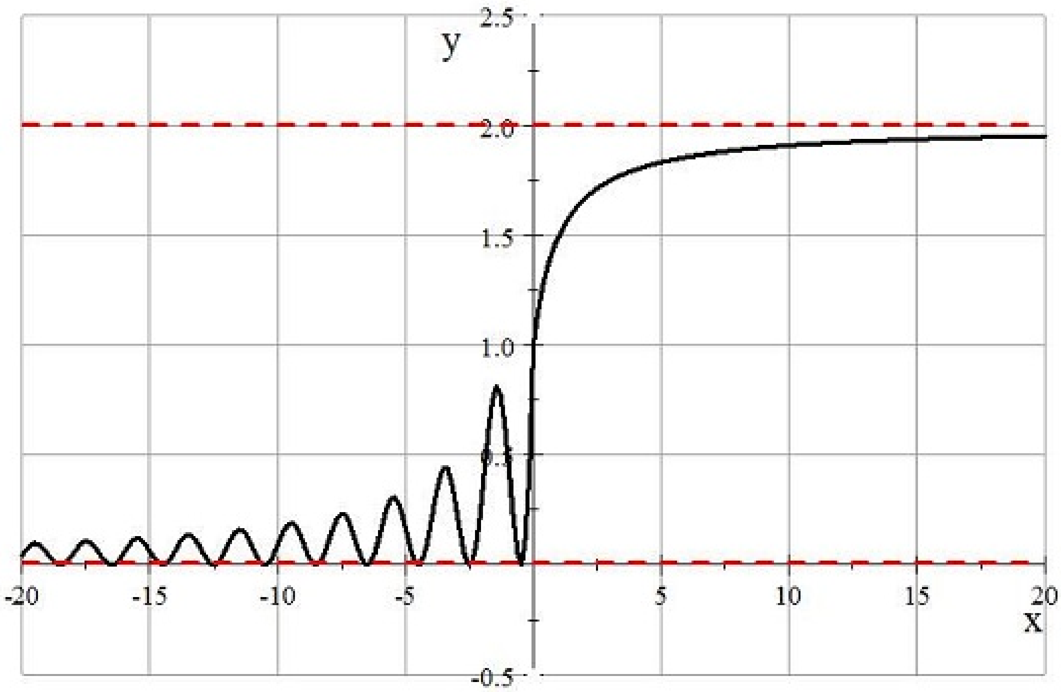
Part (a): Does f have any removable discontinuities? If so, what is the limit of the outputs of f when we allow x to approach these values?

Part (b): Does f have any non-removable discontinuities? If so, what is the limit of the outputs of f when we allow x to approach these values from the left or from the right?

A function f has a *horizontal asymptote* if the output values of f approach a fixed value at the input values get more and more positive or more and more negative.

For example, the figure below shows the graph of the function f defined by the output formula

$$f(x) = \begin{cases} \frac{1+\sin(\pi x)}{1-x} & \text{if} \quad x \leq 0 \\ 1 + \frac{x}{1+x} & \text{if} \quad x > 0 \end{cases}$$



As the graph shows, the function f has two horizontal asymptotes.

- As the input values become more and more positive, the output values of f approach the fixed number $y = 2$.
- As the input values become more and more negative, the output values of f approach the fixed number $y = 0$.

We can use limiting processes to analyze functions to see if they have horizontal asymptotes. When we use limiting processes for this purpose, however, we do not consider input values approaching a fixed number like we did in the previous limit examples. Instead, we consider limiting processes where the input value is allowed to become more and more positive, or more and more negative. The end result of such a limiting process is called a *limit at infinity*. For the function f defined by the output formula above, the graph tells us that

$$\lim_{x \longrightarrow -\infty} f(x) = 0 \qquad\qquad \lim_{x \longrightarrow +\infty} f(x) = 2$$

Often, we can only determine the horizontal asymptotes for a function (if any exist) by sketching the graph of the function. However, if the output formula for the function is the ratio of two functions, then we can do a little more.

Example 3 *Without sketching its graph, determine the horizontal asymptote for the function*

$$f(x) = \frac{2x^2 - 5x - 1}{5x^2 + 4}$$

Solution. This might seem like a daunting task, but we can make the problem a lot simpler if we remember one thing — horizontal asymptotes tell us something about the *very long term* behavior of the function f. Now, as x grows more and more positive, the highest power term in the numerator and denominator will *eventually* dominate. In other words, when the values of x become really large, the squared terms in both the numerator and the denominator will be so much bigger than the other terms that we can *essentially ignore them.*

$$\text{When the value of } x \text{ is VERY large, } \frac{2x^2 - 5x - 1}{5x^2 + 4} \approx \frac{2x^2}{5x^2} = \frac{2}{5} = 0.4$$

The table below shows that the outputs of f do indeed approach the fixed number $2/5$ as the input values get larger and larger.

Value of x	1	100	1000	$10,000$	$100,000$
Value of $f(x)$	$-0.4444...$	0.3899488	0.39899948	0.3998999948	0.3999899999

We may conclue that f has a horizontal asymptote at $y = 2/5$. Using a limiting process to express this fact, we would write

$$\lim_{x \longrightarrow +\infty} f(x) = \frac{2}{5}$$

****

Problem 12. In the example above, would it make any difference if we let x become more and more negative instead? Explain your thinking.

Problem 13. Using the thinking from the previous example, evaluate $\displaystyle\lim_{x \longrightarrow -\infty} \frac{4x^3 + 3}{x^3 - 6x^2 + 5x - 100}$.

Problem 14. Consider the function $f(x) = \dfrac{x^2 - 1}{3x}$.

Part (a): Using the thinking from the previous example, what happens to the output values of f as the input values become more and more positive?

Part (b): Using the thinking from the previous example, what happens to the output values of f as the input values become more and more negative?

Part (c): Does the function f have a horizontal asymptote? Explain your thinking.

Problem 15. Consider the function $f(x) = \dfrac{x^2}{4x^3 - 1}$.

Part (a): Using the thinking from the previous example, what happens to the output values of f as the input values become more and more positive?

Part (b): Using the thinking from the previous example, what happens to the output values of f as the input values become more and more negative?

Part (c): Does the function f have a horizontal asymptote? Explain your thinking.

Problem 16. Evaluate the following limits at infinity. What do the end results of these limiting processes tell you about the horizontal asymptotes of the functions?

(a) $\lim\limits_{x \longrightarrow -\infty} f(x)$, where $f(x) = \dfrac{3x - 5}{x^2}$

(b) $\lim\limits_{y \longrightarrow +\infty} g(y)$, where $g(y) = \dfrac{3y^4 - 7y + 9}{2 - y^4}$

(c) $\lim\limits_{a \longrightarrow -\infty} h(a)$, where $h(a) = \dfrac{a^3}{a^2 - 3a + 1}$

HOMEWORK: Section 2.6 pp. 137 - 138 Problems 3, 4, 5, 7, 13, 15, 17, 18