Let's think about the average value function $y = g_a(h)$ for the function $y = f(x) = \frac{1}{2x}$ that we constructed earlier. For any fixed input value $x = a$, we know that

$$g_a(h) = \frac{f(a+h) - f(a)}{h} = \frac{1}{h}\left[\frac{1}{2(a+h)} - \frac{1}{2a}\right]$$

gives the average rate of change for the function f as the input values change from $x = a$ to $x = a + h$. Furthermore, we used algebra to show that, if we let

$$G_a(h) = -\frac{1}{2a(a+h)}$$

then the output of the function g_a is the same as the output of the function G_a for every input value of h *except* for $h = 0$. The function g_a has no output when $h = 0$, but the function G_a *does* have output when $h = 0$. In fact, we know

$$G_a(0) = -\frac{1}{2a^2}$$

Let's think about what this means for a moment. To help us do this, let's get specific about the fixed value of x so we can compare outputs for the two functions. Let $x = 3$. (You can choose any fixed value for x that you like.) We are now considering the two functions

$$g_3(h) = \frac{1}{h}\left[\frac{1}{2(3+h)} - \frac{1}{6}\right] \qquad G_3(h) = -\frac{1}{6(3+h)}$$

We know that the function g_3 is undefined when $h = 0$, and we know that

$$G_3(0) = -\frac{1}{18} \approx -0.05556$$

While it is true that the function g_3 is undefined when $h = 0$, it is also true that this function *is* defined for values of h that are very close to 0. Let's look at a table of output values for the function g_3 corresponding to input values of h that are very close to 0.

Value of h	-0.015	-0.010	-0.005	-0.001	0	0.001	0.005	0.010	0.015
Value of $g_3(h)$	-0.05583	-0.05574	-0.05565	-0.05557	ERROR	-0.05554	-0.05546	-0.05537	-0.05528

As the values of h get closer and closer to 0, the values of $g_3(h)$ get closer and closer to the value of $G_3(0)$. The values of $g_3(h)$ become better and better approximations to $G_3(0)$ as the values of h get even closer to 0. (You can check this on your calculator.)

Even though the function g_3 is not defined when $h = 0$, the *process* of choosing values of h closer and closer to 0 tells us that the output of the function g_3 gets closer and closer to the value $G_3(0)$. In mathematics, we use a special notation to indicate this.

$$\lim_{h\to 0} g_3\,(h) = \lim_{h\to 0} \frac{1}{h}\left[\frac{1}{2(3+h)} - \frac{1}{6}\right] = -\frac{1}{18}$$

We read this notation as "*the limiting process for the values of* $g_3(h)$ *as the values of* h *approach 0 produces a value of* $-1/18$."

It is customary to shorten this reading to "*the limit of the function* g_3 *as* h *approaches 0 is equal to* $-1/18$."

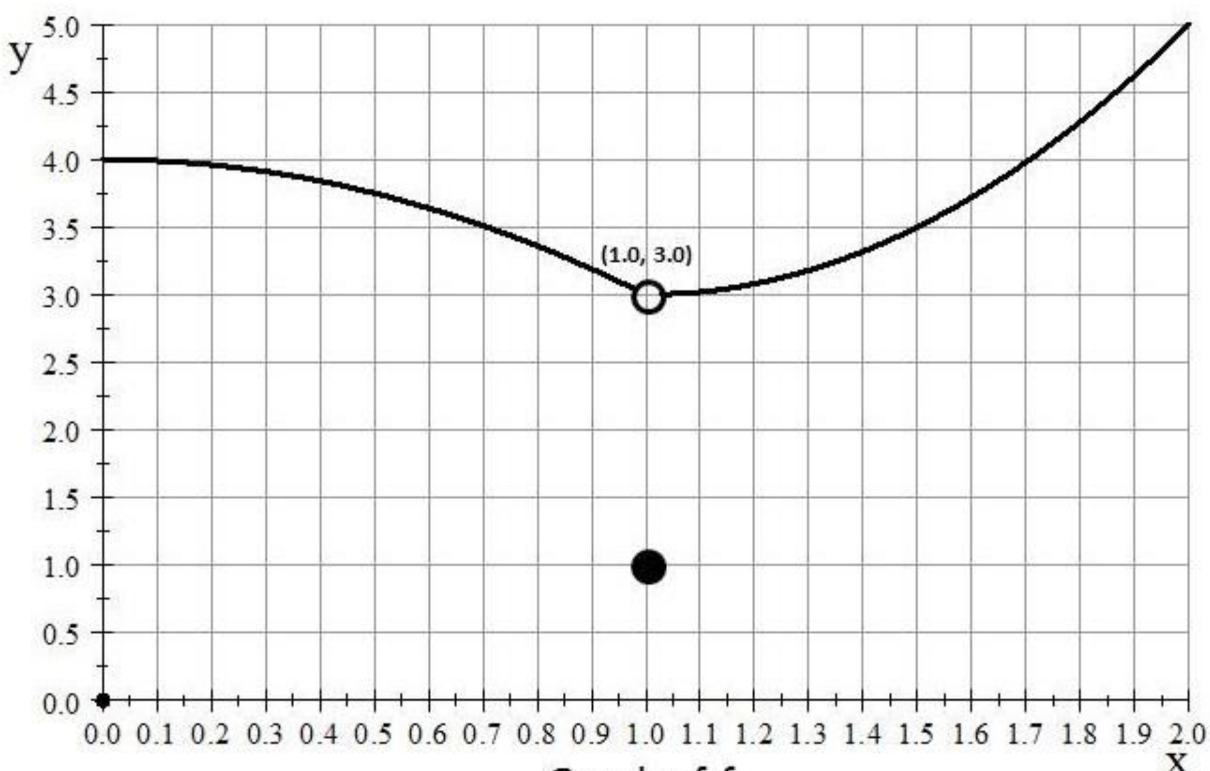
We can apply limiting processes to any function.

Suppose that $y = f(x)$ is a function, and suppose that L is a real number. When we write

$$\lim_{x \to a} f(x) = L$$

we mean that the values of $f(x)$ get closer and closer to L as the values of x get closer and closer to a.

Consider the graph of the function $y = f(x)$ shown in the diagram below.

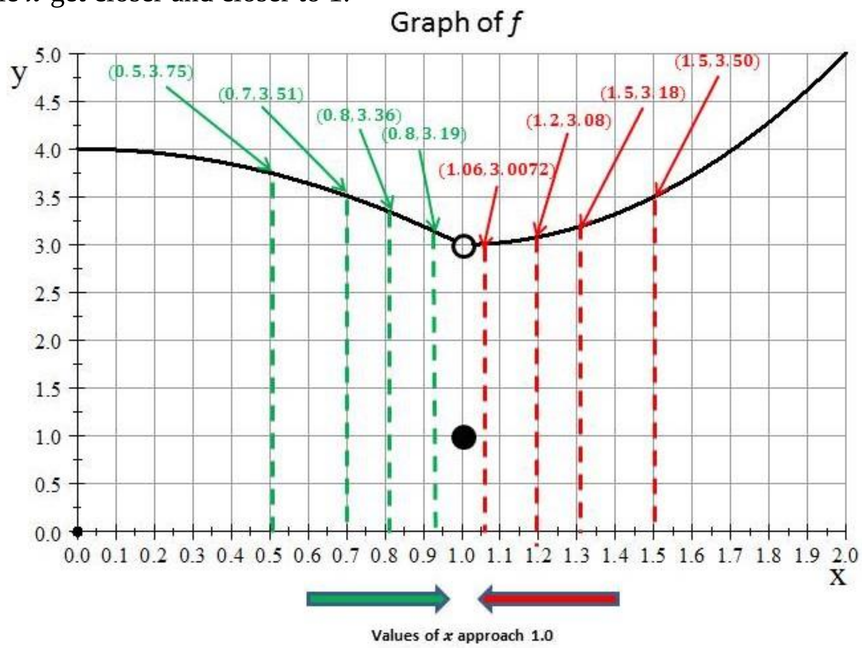


Graph of f

Based on this diagram, what can we say about the process

$$\lim_{x \to 1} f(x)$$

To answer this question, we have to think about what is happening to the values of $f(x)$ as the values of the input variable x get closer and closer to 1.



Even though the diagram shows us that $f(1.0) = 1$, the limiting process produces a different value. In particular,

$$\lim_{x \to 1} f(x) = 3.0$$

The value of a limiting process does not depend on the output value of the function at the limiting input value. Indeed, in the opening example, we know that $g_3(0)$ does not exist, yet

$$\lim_{h \to 0} g_3(h) = -\frac{1}{18}$$

Furthermore, in the last example, we know $f(1.0) = 1$, but we also know that

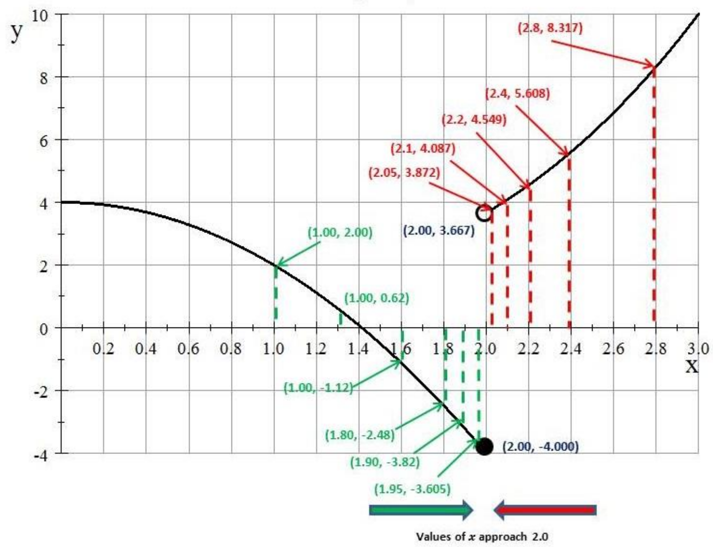
$$\lim_{x \to 1} f(x) = 3.0$$

The value of a limiting process depends only on the *behavior* of the output values for the function as the input values get closer and closer to the limiting input value.

Now, consider the function $y = f(x)$ whose graph is shown in the diagram below. In this case, what can we say about

$$\lim_{x \to 2.00} f(x)?$$

Graph of f



In this case, if we let the input values approach $x = 2.00$ *only from the left*, we can see that the corresponding output values approach -4.000. However, if we let the input values approach $x = 2.00$ *only from the right*, we can see that the corresponding output values approach 3.667. Since the output values are not approaching the same number, we must conclude that

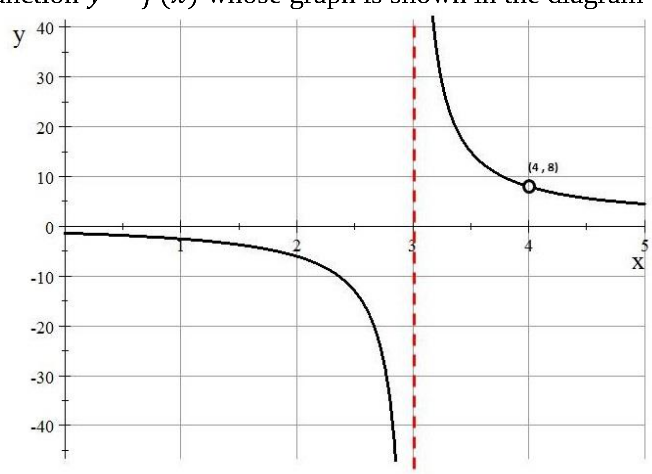
$$\lim_{x \to 2.00} f(x) \quad DOES\ NOT\ EXIST$$

It is possible to say more than this, however. There is a clear pattern to the output values when we let the input values approach $x = 2.00$ *only from the left* and also when we let the input values approach $x = 2.00$ *only from the right*. It is customary to indicate this using the notation

$$\lim_{x \to 2.00^-} f(x) = -4.000 \qquad\qquad \lim_{x \to 2.00^+} f(x) = 3.667$$

Note the use of the "$-$" and the "$+$" superscripts in the limiting processes above.

Now, consider the function $y = f(x)$ whose graph is shown in the diagram below.



Graph of f

It should be clear that, as the input values for f approach $x = 4$, the corresponding output values approach $y = 8$ (even though the function itself is not defined when $x = 4$). Consequently, we may write

$$\lim_{x \to 4} f(x) = 8$$

Now, consider what happens when we allow the input values to approach $x = 3$ *only from the right*. In this case, the corresponding output values do not approach any real number --- the output values keep getting larger and larger. In order to express this behavior symbolically, we write

$$\lim_{x \to 3^+} f(x) = +\infty$$

It is important to realize that this limiting process does not produce a real number --- the limit does not exist. We use the symbol "$+\infty$" merely to indicate that the output values of f get larger and larger as the input values approach $x = 3$ *only from the right*.

On the other hand, when we allow the input values to approach $x = 3$ *only from the left*, the corresponding output values also do not approach any real number --- the output values keep getting more and more negative. In order to express this behavior symbolically, we write

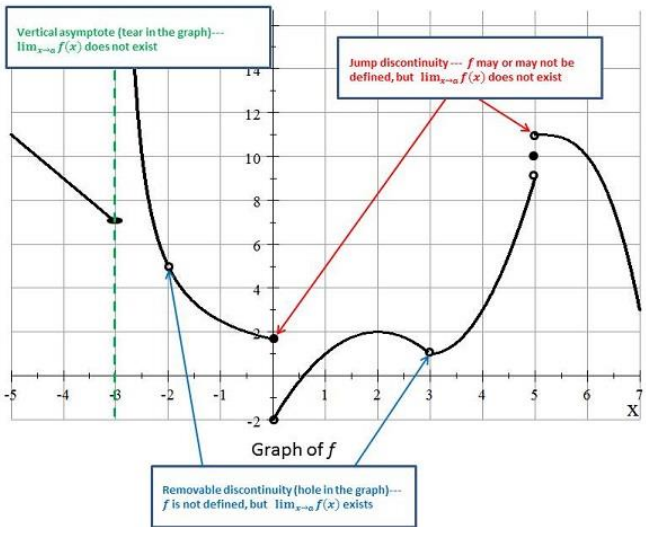
$$\lim_{x \to 3^-} f(x) = -\infty$$

Suppose that $y = f(x)$ is a function. We say that the function f is *continuous* at the input value $x = a$ provided the following conditions are met:

1. The function f is defined at $x = a$; that is, the output value $f(a)$ exists.
2. The limiting process as the values of x approach $x = a$ produces the value $f(a)$; that is,

$$\lim_{x \to a} f(x) = f(a)$$

If the function f is not continuous at an input value $x = a$, then we say that the function f has a *discontinuity* at the value $x = a$. There are three types of discontinuity that we commonly encounter, and they are described in the diagram below.



- *Jump discontinuity* --- When the function f has a jump discontinuity at the input value $x = a$, then
$$\lim_{x \to a} f(x)$$
does not exist, but the limits from the left or right usually do exist. The function may or may not be defined at $x = a$.

- *Vertical asymptote* --- When the function f has a vertical asymptote at the input value $x = a$, then there is a "tear" in the graph at this input value. The function f may or may not be defined at the input value $x = a$, but it will be the case that at least one of the following is true:
$$\lim_{x \to a^-} f(x) = \pm\infty \quad OR \quad \lim_{x \to a^+} f(x) = \pm\infty$$

- *Removable discontinuity* --- When the function f has a removable discontinuity at the input value $x = a$, then there is a "hole" in the graph at this input value. The function f may or may not be defined at the input value $x = a$, but it will be the case that
$$\lim_{x \to a} f(x)$$
exists. If $f(a)$ happens to exist, it will also be the case that
$$\lim_{x \to a} f(x) \neq f(a)$$

HOMEWORK: Section 2.2 (Page 92) Problems 4, 5, 7, 8, 9