Let $y = f(x)$ be a function. In order to determine the value of the limiting process

$$\lim_{x \to a} f(x)$$

(or whether or not the limiting process even produces a value), we must be able to discern patterns in the output values as the input values approach $x = a$. This is often impossible unless we have the graph of the function or have a sufficiently detailed table of values. However, in the special case where the output values of the function f are constructed from the input values solely through algebra, it is often possible to evaluate the limiting process without resorting to tables or graphs.

> We say that a function $y = f(x)$ is *algebraic* provided the output values for f are constructed directly from the input values through addition, subtraction, multiplication, division, or raising to rational powers.
>
> When a function is not algebraic, we say that it is *transcendental*.

The trigonometric functions, exponential functions, and logarithmic functions are all transcendental, because it is not possible to construct the output values of these functions *directly* from their input values using only basic algebra tools.

Here are some examples of algebraic functions:

- *Constant functions* --- functions like
$$y = f(x) = 97 \qquad u = g(t) = \pi$$
- *Power functions* --- functions like
$$y = f(x) = \sqrt{x} \qquad u = g(t) = t^3 \qquad w = h(b) = b^{3/5}$$
- *Polynomial functions* --- functions like
$$y = f(x) = x \qquad u = g(t) = 2t^3 - \sqrt[3]{4}\,t + 8 \qquad w = h(b) = \frac{1}{2}b^4 - 5$$
- *Rational functions* --- functions that are ratios of polynomials like
$$y = f(x) = x^2 - 1 \qquad u = g(t) = \frac{2t^3 - \sqrt[3]{4}\,t + 8}{t - 3} \qquad w = h(b) = \frac{\pi}{b}$$

Any algebraic combination of these types of functions will also be an algebraic function. For example, the following function would be considered algebraic.

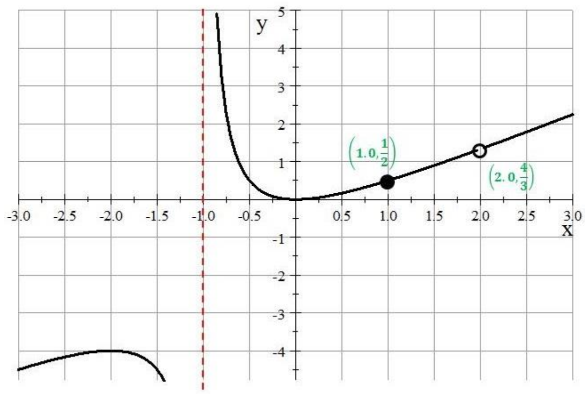
$$y = f(x) = \frac{\left(3 - x^{1/4}\right)^2}{\sqrt[5]{x^3 - 5x}} + 8x$$

The diagram below shows the graph of the rational function

$$y = f(x) = \frac{x^3 - 2x^2}{x^2 - x - 2}$$

Use this graph to determine the value of the following limiting processes.

$$(A)\ \lim_{x\to 1} f(x) \qquad (B)\ \lim_{x\to -1^+} f(x) \qquad (C)\ \lim_{x\to 2} f(x)$$



Looking at the graph, it is clear that the function f has a vertical asymptote at the input value $x = -1$ and has a removable discontinuity at the input value $x = 2$. It is also clear that the graph of the function f has no jump, hole, or tear at the input value $x = 1$. Consequently, the function f is continuous at the input value $x = 1$, but has a discontinuity at the input values $x = -1$ and $x = 2$. From the graph, it is clear that

$$(A)\ \lim_{x\to 1} f(x) = \frac{1}{2} = f(1) \qquad (B)\ \lim_{x\to -1^+} f(x) = +\infty \qquad (C)\ \lim_{x\to 2} f(x) = \frac{4}{3}$$

Now, let's take a closer look at the formula for the function f. Observe that we can factor the numerator and denominator to obtain

$$f(x) = \frac{x^3 - 2x^2}{x^2 - x - 2} = \frac{x^2(x-2)}{(x+1)(x-2)} = \frac{x^2}{x+1} \qquad (x \neq 2)$$

Notice that, prior to simplifying, the roots of the denominator are $x = -1$ and $x = 2$. These are also the input values where the function f has its discontinuities. It turns out that this is no accident.

When a rational function $y = f(x)$ is written as a single fraction, the input values where the function f is discontinuous are precisely the roots of the denominator before simplifying. The function f will be continuous at any other input value.

We can say more than this. Observe that $x = -1$ is a root of the denominator in the function f before simplifying and *is still a root of the denominator after simplifying*. It is also true that $x = -1$ is the only input value where the function f has a vertical asymptote. Once again, this is not an accident.

> Suppose that $y = f(x)$ is a rational function whose formula is written as a single fraction. The function f will have a vertical asymptote at an input value $x = a$ precisely when $x = a$ is a root of the denominator after cancelling common factors.

On the other hand, observe that $x = 2$ is a root of the denominator in the function f prior to simplifying, but *is not a root of the denominator after simplifying*. The process of simplification has *removed* the factor $x - 2$ from the denominator. It is no accident that the function f has a *removable* discontinuity at the input value $x = 2$.

> Suppose that $y = f(x)$ is a rational function whose formula is written as a single fraction. The function f will have a removable discontinuity at an input value $x = a$ precisely when $x = a$ is a root of the denominator *before* cancelling common factors, but is *not* a root of the denominator *after* cancelling common factors.

There is even more we can say. Suppose we let

$$y = g(x) = \frac{x^2}{x+1}$$

We know that $f(x) = g(x)$ for every input value x *except for* $x = 2$. The function f has a discontinuity at $x = 2$, while the function g is continuous at the input value $x = 2$. Consequently, we may conclude

$$\lim_{x\to 2} f(x) = \lim_{x\to 2} g(x) = g(2) = \frac{4}{3}$$

Example 1. Consider the rational function

$$u = f(t) = \frac{3t^3 - 6t}{t^3 - 2t^2 + t}$$

Without sketching the graph of the function f, determine the values of the following limiting processes, if they exist.

$$(A)\ \lim_{t\to 3} f(t) \qquad (B)\ \lim_{t\to 0^+} f(t) \qquad (C)\ \lim_{t\to 1^-} f(t)$$

Solution. First, observe that since

$$3^3 - 2\cdot 3^2 + 3 \neq 0$$

we know that the input value $t = 3$ is not a root of the denominator in the function f. Consequently, the function f is continuous at the input value $t = 3$, and we may conclude

$$\lim_{t \to 3} f(t) = f(3) = \frac{3 \cdot 3^3 - 6 \cdot 3}{3^3 - 2 \cdot 3^2 + 3} = \frac{21}{4}$$

On the other hand, a quick check shows that the input values $x = 0$ and $x = 1$ are both roots of the denominator in the function f. Consequently, the function f will have a discontinuity at both of these input values. In order to determine the value (if any) for the remaining two limit processes, we will have to factor the numerator and denominator of the function f and simplify. Observe

$$f(t) = \frac{3t^3 - 6t}{t^3 - 2t^2 + t} = \frac{3t(t-1)(t+1)}{t(t-1)(t-1)} = \frac{3(t+1)}{t-1} \qquad (t \neq 0)$$

We see that the function f has a removable discontinuity at the input value $x = 0$ and has a vertical asymptote at the input value $x = 1$. Now, observe

$$\lim_{t \to 0^+} f(t) = \lim_{t \to 0^+} \frac{3(t+1)}{t-1} = \frac{3(0+1)}{0-1} = -3$$

It is worth noting that, since the function f has a *removable* discontinuity at the input value $x = 0$, the left-hand limiting process would also produce the value -3.

Since the function f has a vertical asymptote at the input value $x = 1$, we know that the limit

$$\lim_{t \to 1^-} f(t)$$

will be infinite. The only question is, "Will the limiting process lead to $-\infty$ or to $+\infty$?"

We could use the graphing calculator to sketch the graph of the simplified version of the formula for the function f and use the graph to answer this question. However, we can answer the question without referring to the graph at all.

Consider this --- if we are allowing the values of t to approach $t = 1$ *from the left*, then we know that the values of t will always be a little *smaller* than 1. With this in mind, when the values of t are very close to, but just a little *smaller* than 1, we know

$$t - 1 < 0 \qquad AND \qquad 3(t+1) > 0$$

Therefore, we know that the ratio

$$\frac{3(t+1)}{t-1} < 0$$

Consequently, we may conclude that

$$\lim_{t \to 1^-} f(t) = -\infty$$

A function $y = f(x)$ is *piece-wise defined* whenever the rule for f is constructed from several sub-rules, each valid only on a portion of the function's domain. If the function is defined from formulas, then the collection of formulas is grouped together vertically using brackets; and the subsets of the domain on which each formula is valid are placed just to the right of each formula.

Here is an example of a piece-wise defined rational function. (We consider this function to be rational, because each subformula is a rational function.)

$$f(x) = \begin{cases} -x^2 & \text{if} \quad x < 0 \\ 5 & \text{if} \quad 0 \leq x < 2 \\ 3x - 1 & \text{if} \quad 2 < x \end{cases}$$

Piece-wise defined functions are the only common source of jump discontinuities. Jump discontinuities will often occur at the *branch points* where one subformula ceases to be valid and another formula becomes valid. For example, observe that the input value $x = 0$ is a branch point for the function; and observe that

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} x^2 = 0 \qquad\qquad \lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} 5 = 5$$

Since the left-hand and right-hand limiting processes as the input values approach $x = 0$ do not produce the same result, we know that the function f has a jump discontinuity at the input value $x = 0$.

On the other hand, the input value $x = 2$ is also a branch point for the function f. However, in this case we have

$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} 5 = 5 \qquad\qquad \lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} (3x - 1) = 5$$

Since the left-hand and right-hand limiting processes as the input values approach $x = 2$ produce the same result, we know that the function f does not have a jump discontinuity at the input value $x = 2$. Nonetheless, the function f does have a discontinuity at $x = 2$, because there is no output for f defined when $x = 2$. The function f has a removable discontinuity at the input value $x = 2$.

Since there is no opportunity for division by 0 in any of the subformulas for the function f, we may conclude that the function f is continuous at all other input values in its domain.

Homework: Section 2.3 (Pages 102 – 103) Problems 11, 12, 15, 20, 23, 24, 33, 50, 52
Section 2.5 (Page 125) Problems 19, 22, 41, 43