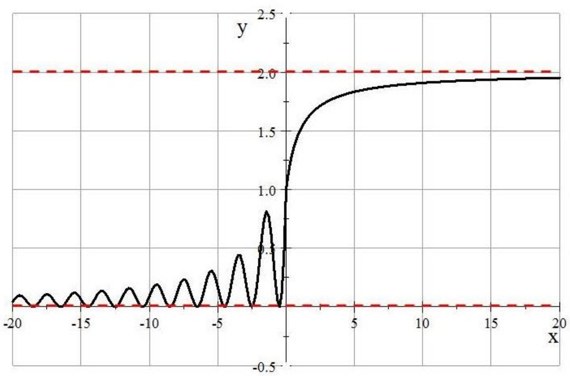
It is also possible to use limiting processes to help describe the *long-term* behavior of output values for functions; that is, the behavior of a function's output as the input values become more and more positive, or more and more negative. Consider the function $y = f(x)$ whose graph is shown below.

$$f(x) = \begin{cases} \dfrac{1+\sin \pi x}{1-x} & \text{if} \quad x \le 0 \\ 1 + \dfrac{x}{1+x} & \text{if} \quad 0 < x \end{cases}$$



As the values of the input variable x become more and more positive, it appears that the output values of the function f are becoming closer to the number 2.0. This becomes even more apparent when we consider a table of values.

Input value x	10	100	1000	10,000	100,000
$f(x)$	1.90909…	1.99009900…	1.999000999000…	1.9999000099990000…	1.99999000009999900000…

We can use limit notation to describe this behavior symbolically. It is customary to write

$$\lim_{x \to +\infty} f(x) = 2$$

This notation represents a different kind of limiting process from that we considered in the previous two lectures. This process, known as a *limit at infinity*, describes the long-term behavior of the output values for the function f as the input values become more and more positive.

Visually, it appears that the graph of the function f is approaching the graph of the horizontal line $y = 2$. We call this line a *horizontal asymptote* for the function f.

Notice that there is a different long-term behavior for the output values of the function f when we let the values of the input variable become more and more negative.

Input value x	-10	-100	-1000	-10,000	-100,000
$f(x)$	0.0909…	0.0099009900…	0.999000999000…	0.000999000999…	0.0000999900009999…

In this case, it appears we would be justified in writing

$$\lim_{x \to -\infty} f(x) = 0$$

The function f has two horizontal asymptotes. The graph of the function f approaches the horizontal line $y = 2$ as the input values grow more and more positive, and the graph of the function f approaches the horizontal line $y = 0$ as the input values grow more and more negative.

Determining the end result of limits at infinity is often impossible without considering the graph of a function; however, there are certain classes of functions whose formulas provide enough information to determine limits at infinity without graphing.

- If $f(x) = ax^r$ for any positive rational number r and nonzero positive constant $a,$ then
$$\lim_{x\to+\infty} f(x) = +\infty$$
Functions of this type do not have horizontal asymptotes as the values of x grow more and more positive.

- If $f(x) = ax^{-r}$ for any positive rational number r and nonzero constant a, then
$$\lim_{x\to+\infty} f(x) = 0$$
Functions of this type have $y = 0$ as a horizontal asymptote as the values of x grow more and more positive.

Example 1. Determine whether or not the function below has a horizontal asymptote without graphing the function.

$$f(x) = \frac{x}{3}$$

Solution. First, note that $x = x^1$, so we know that $f(x) = \frac{1}{3} \cdot x^1$. We may conclude

$$\lim_{x\to+\infty} f(x) = +\infty$$

Now, the function f is also defined for any *negative* input value, so we need to consider what can be said about

$$\lim_{x\to-\infty} f(x)$$

One way to handle this without resorting to the graph of the function is to notice that, *if* we assume $x > 0$, then $-x < 0$. Consequently,

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \frac{1}{3} \cdot x^1 = \lim_{x\to+\infty} \frac{1}{3} \cdot (-x)^1 = \lim_{x\to+\infty} \left(-\frac{1}{3}\right) x^1 = -\infty$$

Therefore, the function f has no horizontal asymptotes.

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Example 2. Evaluate the limit at infinity below without graphing.

$$\lim_{x\to-\infty} 4x^{2/3}$$

Solution. In this case, it is helpful to remember that $x^{2/3} = \sqrt[3]{x^2} = (x^2)^{1/3}$. Observe that

$$\lim_{x\to-\infty} 4x^{2/3} = \lim_{x\to-\infty} 4\cdot\sqrt[3]{x^2} = \lim_{x\to+\infty} 4\cdot\sqrt[3]{(-x)^2} = \lim_{x\to+\infty} 4\cdot\sqrt[3]{x^2} = \lim_{x\to+\infty} 4\cdot x^{2/3} = +\infty$$

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Example 3. Evaluate the limit at infinity shown below without graphing.

$$\lim_{x\to-\infty} \frac{3}{x^5}$$

Solution. Observe that

$$\lim_{x\to-\infty} \frac{3}{x^5} = \lim_{x\to-\infty} 3\cdot x^{-5} = \lim_{x\to+\infty} 3\cdot(-x)^{-5} = \lim_{x\to+\infty} (-3)\cdot x^{-5} = 0$$

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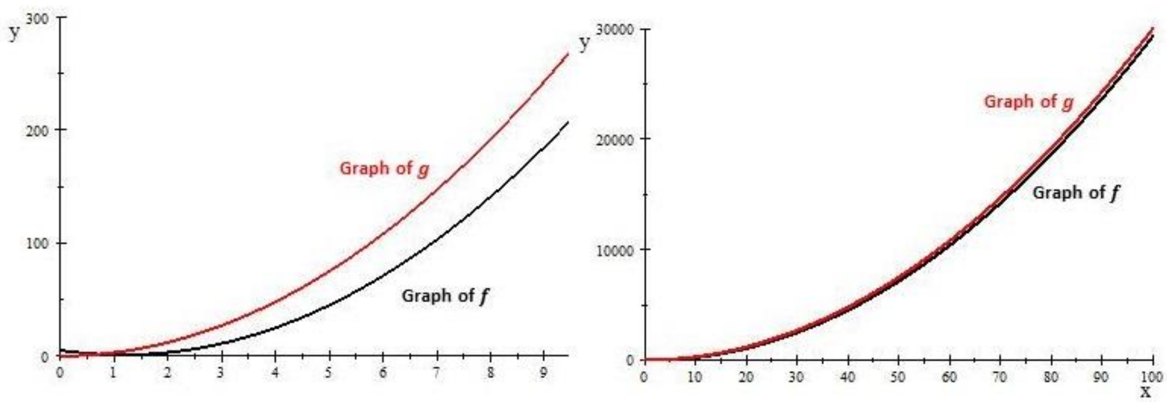
It is also possible to determine the long-term behavior of the output values for other, more complex algebraic functions without resorting to their graphs. The key to accomplishing this task lies in a peculiar feature that certain algebraic functions display as their input values grow more and more positive, or more and more negative --- their output values come to be "dominated" by the output from their highest power term. To see what is meant by this, consider the following two functions.

$$y = f(x) = 3x^2 - 7x + 9 \qquad \text{and} \qquad y = g(x) = 3x^2$$

It is easy to see that these functions are *not* equal --- their output values are different for all but one input value (namely $x = 9/7$). However, compare the graphs of these functions as the values of x grow more and more positive.



As the values of the input variable x grow more and more positive, the graphs of the two functions look more and more alike. The relationship becomes more apparent if we compare the *ratio* of the function outputs for very large values of x.

Input value x	10	100	1000	10,000	100,000
$f(x)$	239	29,309	2, 993,009	299,930,009	29,999,300,009
$g(x)$	300	30,000	3,000,000	300,000,000	30,000,000,000
$f(x)/g(x)$	0.796…	0.976966…	0.997669666…	0.99976669666…	0.9999766669666…

As the values of the input variable grow more and more positive, the contributions from the lower-powered terms in the formula for the function f become less significant than the contribution from the highest power term. This means that, *in the limiting process* as the values of x grow, the functions f and g *eventually behave the same way*. We can use this behavior to our advantage.

Example 4. Determine whether or not the function below has a horizontal asymptote without graphing the function.

$$f(x) = \frac{\sqrt{x}}{1 + 2\sqrt{x}}$$

Solution. First, note that the function f is not defined when $x < 0$, so we only need to consider long-term behavior for the output values of f as the input values grow more and more positive.

In an infinite limiting process, we can ignore all but the highest-power terms in the numerator and the denominator.

$$\lim_{x\to+\infty} \frac{\sqrt{x}}{1 + 2\sqrt{x}} = \lim_{x\to+\infty} \frac{\sqrt{x}}{2\sqrt{x}} = \frac{1}{2}$$

Therefore, the function f has a horizontal asymptote at the line $y = 1/2$.

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It is very important to note that the expressions

$$\frac{\sqrt{x}}{1 + 2\sqrt{x}} \qquad \text{and} \qquad \frac{\sqrt{x}}{2\sqrt{x}}$$

never produce the same output. Only in the *limiting process* as the values of x grow more and more positive can we say that the behavior of these expressions becomes more and more alike.

Example 5. Evaluate the limit at infinity shown below without graphing.

$$\lim_{x\to-\infty} \frac{\sqrt[4]{1 + 2x^4}}{3x - 5}$$

Solution. Observe that

$$\lim_{x\to-\infty} \frac{\sqrt[4]{1 + 2x^4}}{3x - 5} = \lim_{x\to-\infty} \frac{\sqrt[4]{2x^4}}{3x} = \lim_{x\to-\infty} \frac{|x|\sqrt[4]{2}}{3x} = \lim_{x\to+\infty} \frac{|(-x)|\sqrt[4]{2}}{3(-x)} = \lim_{x\to+\infty} \frac{|x|\sqrt[4]{2}}{-3x} = -\frac{\sqrt[4]{2}}{3}$$

Consequently, we may conclude that

$$\lim_{x\to-\infty} \frac{\sqrt[4]{1 + 2x^4}}{3x - 5} = -\frac{\sqrt[4]{2}}{3}$$

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Homework: Section 2.6 (Pages 137 – 138) Problems 3, 4, 5, 6, 9, 15, 18, 21, 23, 31