TEAM 1

Problem 1. Construct the formula for $r = f'(x)$ if $f(x) = \sqrt{x + \sqrt{x + \sqrt{x}}}$.

Problem 2. The theory of relativity predicts that an object whose mass is M kg when it is stationary will appear to gain mass when it is moving at speeds near that of light. In particular, the mass m of the object (measured in kg) moving at v meters per second in a vacuum is given by

$$m = f(v) = \frac{M}{\sqrt{1 - (v/c)^2}}$$

where c is the speed of light in a vacuum (measured in meters per second). Compute $\frac{dm}{dv}$. What meaning does this derivative have?

Problem 3. The charge Q (measured in coulombs) on a capacitor which starts discharging at time $t = 0$ seconds is given by the formula

$$Q = f(t) = \begin{cases} C & \text{if} \quad t \le 0 \\ C \cdot e^{-t/(RC)} & \text{if} \quad t > 0 \end{cases}$$

The constants R and C are positive constants. The current flowing through the circuit is given by $\frac{dQ}{dt}$. Is the current defined when $t = 0$ seconds? Justify your answer.

Problem 4. Construct the formula for $\frac{d^2y}{dx^2}$ for the expression $x^3 - y^3 = xy$.

Problem 5. Determine all ordered pairs (a, b) where the tangent line to the graph of $6 = xy - y^3$ is either horizontal or undefined.

Problem 6. Determine the antiderivative family for the function $y = f(x) = \tan^2(x)\sec^4(x)$.

Problem 7. Let $x \neq 0$ and consider the functions shown below.

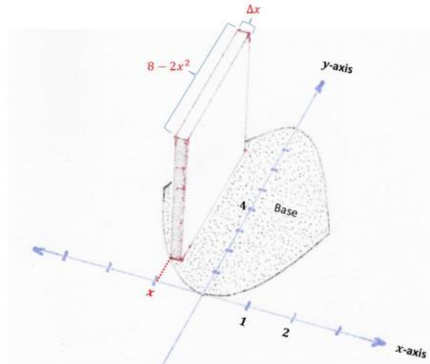
$$I_0(x) = \int_{-1}^{1} \cos(xz)\, dz \qquad I_1(x) = \int_{-1}^{1} (1 - z^2)\cos(xz)\, dz$$

Show that $xI_0(x) = 2\sin(x)$ and $x^3 I_1(x) = 4\sin(x) - 4x\cos(x)$.

Problem 8. Use partial fraction decomposition to determine the antiderivative family for the function defined by the formula

$$f(x) = \frac{3x^2 - x + 1}{x^3 + x}$$

Problem 9. Determine the volume of the solid whose cross-sections perpendicular to the xy-plane and parallel to the y-axis are squares and whose base in xy-plane is the region bounded by the graphs of $y = f(x) = x^2$ and $y = f(x) = 8 - x^2$.



Problem 10. The residents of Raytown are dealing with an epidemic of Creeping Crud. The first residents fell ill on February 12. Let N be the total number of people in Raytown who have contracted Creeping Crud, and let t be the time passed since February 12, measured in days. Suppose we know

$$N = f(t) = \frac{1000}{1 + 75e^{-0.1t}}$$

Part (a). As the epidemic runs its course, what is the total number of people who fall ill? Justify your answer.

Part (b). Is there ever time interval when at least than 15 people fell ill per day? Justify your answer.

Problem 11. Two new houses need to be supplied with telephone lines. The telephone company needs to install a telephone pole immediately adjacent to a road that passes to the south of the two new houses (see the following image). The lines supplied to each house will extend directly from the telephone pole the company needs to install. One house is 56 meters from the road and the other is 29 meters from the road. The horizontal distance (east to west) between the two houses is 48 meters. Where should the telephone company install the pole in order to minimize the total amount of wire needed to supply the two houses with telephone service?

Problem 12. Determine whether or not each of the following infinite series converge. Carefully explain how you arrived at your answer.

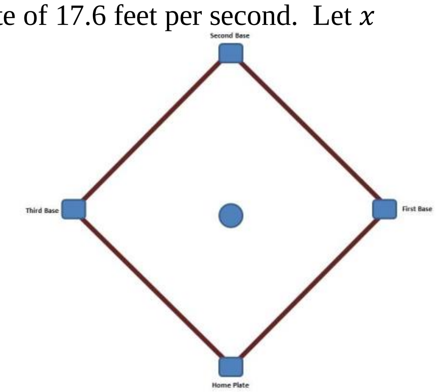
$$(a)\ \sum_{j=0}^{\infty} \frac{2j}{j^2+1} \qquad (b)\ \sum_{m=1}^{\infty} \frac{1+m}{m^3} \qquad (c)\ \sum_{n=2}^{\infty} \frac{\cos(\pi n)}{n}$$

Problem 13. Suppose that a differentiable function $y = f(x)$ has the property that $f(2) = -4$ and $f(4) = -2$. Explain why there must exist some input value in the input interval $2 \le x \le 4$ where $f'(x) = 1$. Can you determine the value of x?

Problem 14. Solve the differential equation $6x^2 - 3yx^2 = \frac{dy}{dx}$.

Problem 15. On average, a baseball player gets a hit in one out of three attempts. Assuming that the attempts are independent, what is the probability that she gets at least one hit in six attempts?

Problem 16. A softball diamond is a square with each side 60 feet in length. At the exact moment the ball is hit, Lauren runs from first base to second base at a constant rate of 17.6 feet per second. Let x represent Lauren's distance from first base, measured in feet, let h represent Lauren's distance from home plate, measured in feet, and let t represent the number of seconds elapsed since the ball was hit.What is the value of $\frac{dh}{dt}$ when Lauren is 21.6 feet from second base?



Problem 17. Consider the matrix shown below. Are there any values of h that will make this matrix invertible?

$$\begin{pmatrix} 0 & 0 & 0 & 2 \\ 2 & 4 & 3 & h \\ 0 & 0 & 1 & h^2 \\ 1 & -1 & 0 & 2h \end{pmatrix}$$

Problem 18. Consider the system of linear equations represented by the matrix equation below. Is this system dependent, independent, or inconsistent? If the system is consistent, determine its solution set.

$$\begin{pmatrix} 1 & 1 & -2 \\ 2 & -1 & -7 \\ 1 & 2 & -1 \end{pmatrix} (x \quad y \quad z)^T = \begin{pmatrix} 1 \\ 8 \\ 0 \end{pmatrix}$$

Problem 19. Molly sits a cup of coffee on her kitchen table. The temperature of the room is a constant 77°F, and the cup of coffee is initially 155°F. Ten minutes later, the coffee is 110°F. Assuming Molly has forgotten about the coffee, what is the temperature of the coffee one hour after Molly sat the cup on the table?

Problem 20. Construct a slope field for the differential equation shown below, then use your slope field to approximate the graph of the solution that passes through the point $(0,3)$.

$$\frac{dy}{dx} = \frac{x}{2 - 2y}$$

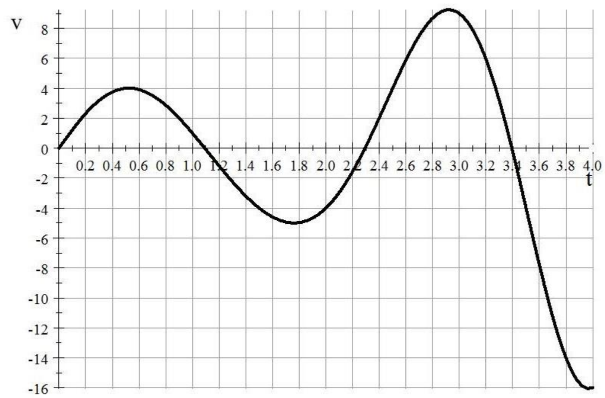
Problem 21. Solve the second-order initial-value problem shown below.

$$8u'' + 2u' = u \qquad u(0) = 2 \quad u'(0) = -1$$

Problem 22. Tiny tyke Tyson is riding his bicycle back and forth along a straight road that ends at the shore of Lake Rheely Whet. Let t represent the number of minutes passed since Tyson started riding his bicycle, and let d represent Tyson's distance from the Lakeshore, measured in feet. The graph below shows Tyson's velocity (measured in feet per minute); assume negative output represents velocities toward the lakeshore.

Part (a). At what time was Tyson the greatest distance from the lake? Can you determine how far he was from the lake at that time?

Part (b). When Tyson started riding his bicycle, was he at the lakeshore?



Problem 23. Suppose that $\mathcal{V} = (V, +, \cdot)$ is a vector space over the scalar field of real numbers with scalar multiplication $\cdot$ and vector addition $+$; and suppose $\mathcal{V}$ has Dimension 8. If W_1 is a subspace of Dimension 5, and W_2 is a subspace of Dimension 4, then what are the possible dimensions of the subspace $W_1 \cap W_2$?

Problem 24. Consider the multivariable function $f(u, v) = \cos(uv) - v$. If $u(x, y) = x\ln(y)$ and $v(x, y) = \cos(x)\sin(y)$, then what is the formula for the following derivatives?

$$\frac{\partial^2 f}{\partial y \partial x} \qquad \frac{\partial^2 f}{\partial x \partial x}$$

Problem 25. Change the order of integration and evaluate the iterated integral

$$\int_0^{\pi/2} \int_x^{\pi/2} \frac{\sin(y)}{y} \, dy dx$$

Problem 26. Find a basis for the null space, row space, and column space of the matrix below.

$$A = \begin{pmatrix} 1 & 3 & 1 & 4 \\ 2 & 7 & 3 & 9 \\ 1 & 5 & 3 & 1 \\ 1 & 2 & 0 & 8 \end{pmatrix}$$

Problem 26. Consider the vector space $\mathcal{R}^4 = (\mathbb{R}^4, +, \cdot)$ endowed with the Euclidean inner product

$$\langle \boldsymbol{u}, \boldsymbol{v} \rangle = \boldsymbol{u}\boldsymbol{v}^T \quad \text{(thinking of } \boldsymbol{u} \text{ and } \boldsymbol{v} \text{ as } 1 \times 4 \text{ matrices)}$$

Construct the projection of the vector $\boldsymbol{u} = (3,2,-1,0)$ onto the vector $\boldsymbol{v} = (1,1,1,1)$ relative to this inner product; that is, determine the vector $\mathrm{Proj}_{\boldsymbol{v}}(\boldsymbol{u})$.

Problem 27. Let $\mathbb{R}$ denote the set of real numbers, let $X = \mathbb{R} - \{0,1\}$, and let CR be the set consisting of the following functions from X to X.

$$\varepsilon(x) = x \qquad q(x) = 1 - x \qquad r(x) = \frac{1}{x} \qquad s(x) = \frac{1}{1-x} \qquad t(x) = \frac{x}{x-1} \qquad u(x) = \frac{x-1}{x}$$

Show that the set CR forms a group with respect to the operation of function composition.

Problem 28. Let $\boldsymbol{\mathcal{Z}}_n = (\mathbb{Z}_n, \boxplus_n)$ represent the group composed of the set $\mathbb{Z}_n = \{0,1,2, \dots, n-1\}$ under the operation $\boxplus_n$ of addition modulo n. Is the bijection $g\colon \mathbb{Z}_3 \times \mathbb{Z}_2 \to \mathbb{Z}_6$ defined by the arrangement below an isomorphism between the groups $\boldsymbol{\mathcal{Z}}_3 \times \boldsymbol{\mathcal{Z}}_2$ and $\boldsymbol{\mathcal{Z}}_6$?

$$(0,0) \xrightarrow{g} 0 \quad (0,1) \xrightarrow{g} 3 \quad (1,0) \xrightarrow{g} 2 \quad (1,1) \xrightarrow{g} 1 \quad (2,0) \xrightarrow{g} 4 \quad (2,1) \xrightarrow{g} 5$$

Problem 29. Let $\mathfrak{Z} = (\mathbb{Z}, +, \cdot)$ denote the ring of integers under integer addition and multiplication. For any fixed positive integer n, let $n\mathbb{Z}$ represent the set of integer multiples of n.

Part (a). Prove that $n\mathbb{Z}$ is an ideal of $\mathfrak{Z}$.

Part (b). Prove that the ideal $2\mathbb{Z}$ is a prime ideal of $\mathfrak{Z}$.

Part (c). Prove that the ideal $4\mathbb{Z}$ is *not* a prime ideal of $\mathfrak{Z}$.

Problem 30. The Heine-Borel Theorem states that a subset U of the set $\mathbb{R}$ of real numbers is *compact* if and only if the set is *closed* and *bounded* with respect to the usual topology on $\mathbb{R}$. Are any of the following subsets of $\mathbb{R}$ compact?

$$U = \bigcap \left\{ \left(-\frac{1}{n}, \frac{1}{n} \right) : n \in \mathbb{Z}^+ \right\} \qquad V = \bigcup \left\{ \left[\frac{1}{n}, \frac{n-1}{n} \right] : n \in \mathbb{Z}^+, n \geq 3 \right\} \cup \{0\}$$

$$W = \left\{ \frac{1}{n} : \; n \in \mathbb{Z}^+ \right\} \cup \{0\}$$

The *modulus* (or *magnitude*) of a complex number $z = a + ib$ is defined to be $\|z\| = \sqrt{a^2 + b^2}$. Thinking of complex numbers as points (a, b) in the plane, the modulus is the distance from the origin to z. The smallest positive angle θ between the positive x-axis and the line segment from $(0,0)$ to (a, b) is called the *argument* of z.

Problem 31. If $z = a + ib$ is a complex number, it is customary to call the component a the *real* part of z and to call b the *imaginary* part of z and denote these components by $\mathrm{Re}(z)$, and $\mathrm{Im}(z)$ respectively. The subsets of complex numbers below represent the graph of a curve. In each case, what is the curve?

$$S = \{z \in \mathbb{C} : \|z\| = 2\mathrm{Re}(z) + 1\} \quad S = \{z \in \mathbb{C} : \|z\| = \mathrm{Im}(z) + 2\}$$

$$S = \left\{ z \in \mathbb{C} : \|z\| = 9 - \frac{\mathrm{Re}(z)}{2} \right\}$$

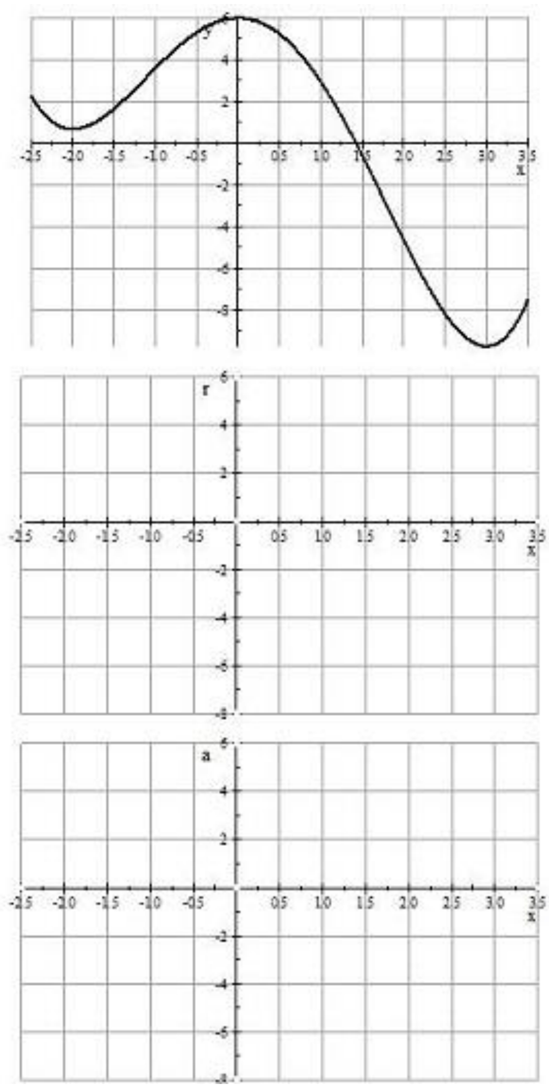
TEAM 2

Problem 1. What is the formula for $\frac{df}{dx}$ if we know $f(u) = u^2 \cos(u)$, and we also know

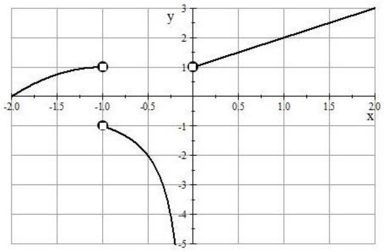
Part (a). $u(x) = 3^x - 2x$

Part (b). $u(x) = x\text{Arctan}(x)$

Problem 2. The trace below shows the graph of a function $y = f(x)$. On the first grid, sketch the graph of the derivative function $r = f'(x)$ for the function f. Use your sketch of the derivative function f' to sketch the second derivative function $a = f''(x)$.



Problem 3. The diagram below shows the graph of the *derivative* function for a function $y = f(x)$. Assume that the function f is continuous.



Part (a). Based on this diagram, at what input values does the function f have a local maximum output?

Part (b). On what input intervals will the *second derivative* function $a = f''(x)$ have positive output?

Problem 4. The function $y = f(t) = A\sin\left(\sqrt{\frac{k}{m}}\,t\right)$ gives the vertical height y of a pendulum bob of mass m above the midpoint in terms of the time t passed since the pendulum began to oscillate. The constant k measures the stiffness of the spring, and A represents the maximum height attained.

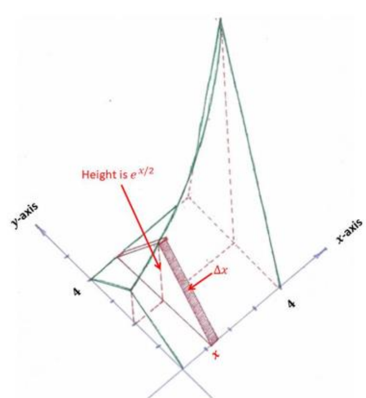
Part (a). At what times does the pendulum bob have maximum acceleration?

Part (b). Construct a formula for the period P of the oscillation and compute $\frac{dP}{dm}$. What does the sign of this derivative tell you?

Problem 5. Construct the formula for $\frac{d^2y}{dx^2}$ for the expression $x\cos(y) = y\cos(x)$.

Problem 6. What is the area of the region enclosed between the curves $y = 1$, $y = 3$, $y = f(x) = x$, and $y = \sqrt{x-3}$?

Problem 7. A solid has a square base of length 4 in the xy-plane. Cross sections of this solid taken perpendicular to the xy-plane and parallel to the y-axis are isosceles triangles whose heights are given by the function $z = f(x) = e^{x/2}$. (See the figure on the right.) Determine the volume of this solid.



Problem 8. Find the volume of the solid obtained by revolving the region enclosed between the curves

$$y = f(x) = \text{Arcsin}(x) \quad y = g(x) = 0 \quad x = \frac{1}{2} \quad \text{and} \quad x = 1$$

around the y-axis.

Problem 9. For nonzero x and any nonnegative integer n, consider the function

$$I_n(x) = \int_{-1}^{1} (1 - z^2)^n \cos(xz)\, dz$$

Use integration by parts twice to show that, for $n \geq 2$, we have

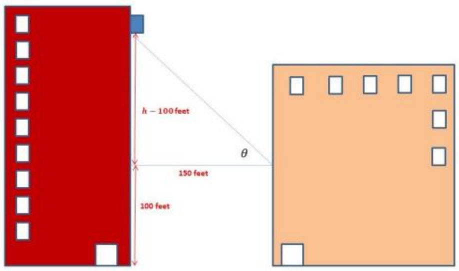
$$x^2 I_n(x) = 2n(2n-1) I_{n-1}(x) - 4n(n-1) I_{n-2}(x)$$

Hint: $\displaystyle\int_{-1}^{1} z^2 (1 - z^2)^{n-2} \cos(xz)\, dz = \int_{-1}^{1} z^2 (1 - z^2)^{n-2} \cos(xz)\, dz + I_{n-2}(x) - I_{n-2}(x)$

Problem 10. Use a partial fraction decomposition to evaluate the definite integral below.

$$\int_{-3}^{-1} \frac{1 + x + 3x^2 - 3x^3}{x(x-1)}\, dx$$

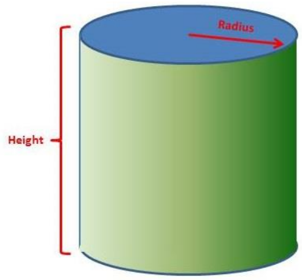
Problem 11. The No-Tell Hotel has an elevator on the outside of the building for the amusement of its guests. Suppose the hotel is 300 feet tall. Lazarus is looking out a window 100 feet off the ground and 150 feet away from the elevator. Lazarus sees the elevator start descending from the top of the hotel at a constant speed of 30 feet per second. Let t represent the time passed in seconds since the elevator began its descent, and let θ represent the radian measure of the angle between Lazarus' line-of-sight to the elevator and the horizontal. (See diagram on the right.)



Part (a). Find the formula that gives the values of θ in terms of the values of t. (You will need the arctangent function.)

Part (b). The expression $\frac{d\theta}{dt}$ is a measure of how fast the elevator appears to be moving for Lazarus. At what height above the ground will the elevator appear to moving the fastest?

Problem 12. Cans R Us needs to construct a closed cylindrical can with a fixed volume of 356 cubic centimeters. What dimensions (radius and height) will minimize the surface area of the can?



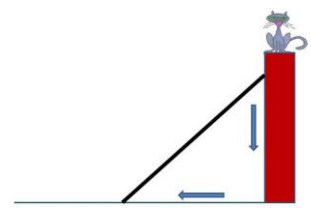
Problem 13. Each of the following series converge. Determine the exact number that each series converges to. Carefully explain how you obtain your answers.

$$(a)\ \sum_{t=2}^{\infty} \frac{4}{3^{2t-1}} \qquad (b)\ \sum_{p=3}^{\infty} \left(\frac{1}{2^p} + \frac{(-1)^{3p-1}}{5^{3p}}\right)$$

Problem 14. Solve the differential equation $xy' + 2y = e^{x^2}$.

Problem 15. A box contains six good and two defective items. If three items are selected at random without replacement, what is the probability exactly one of them will be defective?

Problem 16. Justo has a twelve-foot ladder leaning against a vertical wall as shown below. Mr. Hipsickles scampers up the ladder, causing the base of the ladder to slide away from the wall. Let x represent the horizontal distance, measured in feet, between the base of the ladder and the wall; and let y represent the vertical distance, measured in feet, between the top of the ladder and the ground. Let t represent the number of seconds passed since the ladder began to slide.
When the base of the ladder is six feet from the wall, suppose we know that $\frac{dx}{dt} = 7.6$ feet per second. What is the corresponding value of $\frac{dy}{dt}$?



Problem 17. Use elementary row operations to help compute the determinant of the following matrix. Carefully explain your strategy.

$$\begin{pmatrix} 2 & 2 & 3 & 4 & 1 \\ 1 & 1 & 1 & 1 & 0 \\ 2 & 0 & 1 & 3 & 2 \\ 0 & 1 & 4 & 5 & 2 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

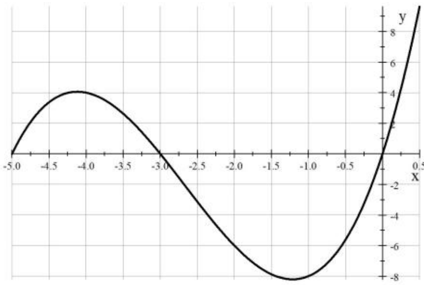
Problem 18. Carefully justify your answer to the following questions.

Part (a). Suppose a system of linear equations contains four equations and seven unknowns. Is it possible for this system to be consistent? Is it possible for this system to have a unique solution?

Part (b). Suppose a system of linear equations contains seven equations and four unknowns. Is it possible for this system to be consistent? Is it possible for this system to have a unique solution?

Problem 19. What is the solution set for the differential equation $y'' - 9y = 0$?

Problem 20. Consider the differential equation $y' = f(x)$, where the graph of the function f is shown below. Use this graph to construct a slope field for this differential equation, then use your slope field to approximate the graph of the solution which passes through the point $(-1,1)$ and the solution which passes through the point $(-4,-1)$.



Problem 21. The population of blue-finned mud twaddlers in Lake Rheely Whet increases at a rate proportional to the number present at any time. In the absence of any outside factors, the population will triple in three weeks. However, each day, fifteen mud-twaddlers swim into the lake, sixteen are eaten by predators, and seven die of natural causes. If there are initially 100 mud-twaddlers, will the population in Lake Rheely Whet survive?

Problem 22. Consider the vector space $\mathcal{R}^4 = (\mathbb{R}^4, +, \cdot)$ with the usual vector addition $+$ and scalar multiplication $\cdot$ over the field of real numbers. Construct a basis for the subspace W spanned by the vectors

$$\begin{pmatrix} -1 \\ 3 \\ 0 \\ 4 \end{pmatrix}, \begin{pmatrix} 2 \\ -3 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \\ 2 \\ 12 \end{pmatrix}, \begin{pmatrix} 3 \\ 0 \\ 3 \\ 12 \end{pmatrix}, \begin{pmatrix} 4 \\ 6 \\ 2 \\ 14 \end{pmatrix}$$

Is it true that $W = \mathbb{R}^4$?

Problem 23. Consider the multivariable function $f(x,y) = \cos(x^2 - 2xy)$. Compute the following partial derivatives.

$$\frac{\partial^2 f}{\partial x \partial y} \qquad \frac{\partial^2 f}{\partial y \partial y} \qquad \frac{\partial^3 f}{\partial y \partial x \partial y}$$

Problem 24. Consider the function $z = f(x,y) = x^2 + y^2 + \sin\left(\frac{\pi x y}{2}\right)$. What is the equation of the tangent plane to this surface at the point $(3,1,9)$?

Problem 25. Consider the vector space $\mathcal{R}^4 = (\mathbb{R}^4, +, \cdot)$, and let $B = \{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3, \boldsymbol{e}_4\}$ be the standard basis for $\mathcal{R}^4$ written as column vectors. Suppose that $T : \mathcal{R}^4 \to \mathcal{R}^3$ is a linear transformation having the property that

$$T(\boldsymbol{e}_1) = \begin{pmatrix} 4 \\ 2 \\ 2 \end{pmatrix} \quad T(\boldsymbol{e}_2) = \begin{pmatrix} 5 \\ 5 \\ 4 \end{pmatrix} \quad T(\boldsymbol{e}_3) = \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix} \quad T(\boldsymbol{e}_4) = \begin{pmatrix} 2 \\ 6 \\ 4 \end{pmatrix}$$

Part (a). Construct the formula for $T[(x,y,z,w)^T]$.

Part (b). What is the dimension of the subspace $T(\mathcal{R}^4)$?

Problem 26. Consider the vector space $\mathcal{R}^3 = (\mathbb{R}^3, +, \cdot)$.

Part (a). Show that the following collection of vectors forms a basis for $\mathcal{R}^3$.

$$S = \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

Part (b). Construct an orthonormal basis for $\mathcal{R}^3$ from this basis relative to the inner product defined by

$$\boldsymbol{u} * \boldsymbol{v} = \boldsymbol{u}^T \boldsymbol{v}$$

Problem 27. Let $\boldsymbol{\mathcal{Z}}_n = (\mathbb{Z}_n, \boxplus_n)$ represent the group composed of the set $\mathbb{Z}_n = \{0,1,2,\ldots,n-1\}$ under the operation $\boxplus_n$ of addition modulo n and consider the product group $\boldsymbol{\mathcal{Z}}_3 \times \boldsymbol{\mathcal{Z}}_4 \times \boldsymbol{\mathcal{Z}}_6$.

Part (a). What is the order of the element $(2,2,2)$ in this group?

Part (b). What is $(1,3,3)^{-8}$ in this group?

Part (c). Construct the subgroup of $\boldsymbol{\mathcal{Z}}_3 \times \boldsymbol{\mathcal{Z}}_4 \times \boldsymbol{\mathcal{Z}}_6$ generated by the set $\{(2,2,2), (0,0,3)\}$.

Problem 28. Let $U_{2,2}$ represent the set of all nonsingular (invertible) 2×2 matrices with real number entries. If we let $*$ represent 2×2 matrix multiplication, then $\boldsymbol{GL_2} = (U_{2\times 2}, *)$ forms a group (called the *General Linear Group*). The identity for this group is the 2×2 identity matrix, and for each $A \in U_{2\times 2}$, we have

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \Rightarrow \quad A^{-1} = \frac{1}{\mathrm{Det}(A)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Prove that the following collection of matrices is a subgroup of $\boldsymbol{GL_2}$. Is this subgroup normal?

$$H = \{A \in U_{2\times 2} : \mathrm{Det}(A) = 1\}$$

Problem 29. Suppose that $\mathcal{R} = (R, +, *)$ is a commutative ring, and let S be a proper ideal of $\mathcal{R}$. Let $a \in R$ be such that $a \notin S$ and let

$$S_a = \{m + a * b : m \in S \text{ and } b \in R\}$$

Part (a). Prove that S_a is an ideal of $\mathcal{R}$.

Part (b). If S is a maximal ideal of $\mathcal{R}$, explain why we must have $S_a = R$.

Problem 30. Let $X = \{a, b, c, d, e\}$ and consider the following topology defined on X.

$$\Omega = \big\{\emptyset, X, \{a,b\}, \{e,a\}, \{a\}, \{a,c\}, \{e,a,b\}, \{c,d\}, \{c,b,e\}, \{c\}, \{b,e\}, \{a,c,d\}, \{a,c,e\}, \{a,b,c,e\}\big\}$$

Part (a). Are there any subsets that are *clopen* (both closed and open) with respect to this topology?

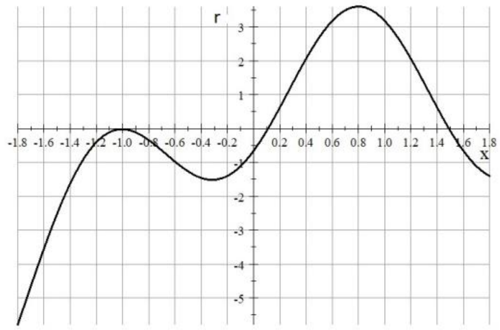
Part (b). What is the topological closure of the set $\{a, b\}$?

The *modulus* (or *magnitude*) of a complex number $z = a + ib$ is defined to be $\|z\| = \sqrt{a^2 + b^2}$. Thinking of complex numbers as points (a, b) in the plane, the modulus is the distance from the origin to z. The smallest positive angle θ between the positive x-axis and the line segment from $(0,0)$ to (a, b) is called the *argument* of z.

Problem 31. Write the complex number $z = 1 + i\sqrt{3}$ in polar form and use this form to determine the value of $\sqrt[3]{z}$.

TEAM 3

Problem 1. The diagram below shows the graph of the derivative function for a function $y = f(x)$.



Graph of $r = f'(x)$

Part (a). On what input intervals is the graph of the function f increasing?

Part (b). On what input intervals is the graph of the function f decreasing?

Part (c). At what input values does the function f have a local minimum output?

Part (d). Suppose we know that $f(0.8) = -1.7$. What is the approximate local minimum output value of f?

Problem 2. Consider the composite function $y = h(x) = (f \circ g)(x)$ where we let $y = f(u) = \sqrt{u - 1}$ and $u = g(x) = x^{-2}$.

Part (a). What is the formula for $r = h'(x)$?

Part (b). Explain why $a = h'(x)$ has no solution when $|a| \leq 1$.

Part (c). Are there any solutions to the equation $4 = h'(x)$?

Problem 3. What is the point-slope formula for the line tangent to the ellipse defined by the formula $(x - 1)^2 + 4(y + 2)^2 = 37$ at the point $(2,1)$?

Problem 4. What is the fourth derivative of the function $y = f(x) = e^x \cos(x)$?

Problem 5. Consider the function defined by the formula below. Is this function differentiable at $r = 0$? Carefully justify your answer.

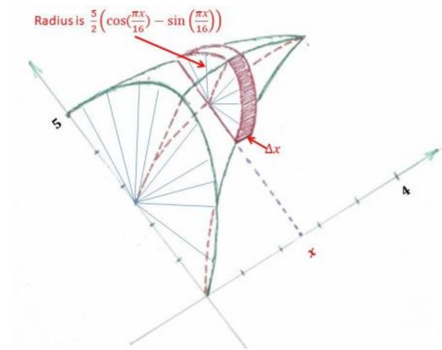
$$y = f(r) = \sqrt{1 + 4r^2 - 4|r|} - 2|r|$$

Problem 6. Consider a closed cylindrical can like the one shown in Team 2 Problem 12. Suppose that the material for the top and bottom of the can costs \$0.75 per square centimeter, while the material for the side costs \$0.50 per square centimeter. Assuming the volume of the can is fixed at 356 cubic centimeters, what dimensions will minimize the cost of constructing the can?

Problem 7. A solid has a base in the xy-plane that is the region enclosed between $x = 0$, $x = 4$, and the curves

$$y = f(x) = 5\cos\left(\frac{\pi x}{16}\right) \quad y = g(x) = 5\sin\left(\frac{\pi x}{16}\right)$$

Cross sections of this solid taken perpendicular to the xy-plane and parallel to the y-axis are half-circles centered on the midline of the base. (See the figure on the right.) Determine the volume of this solid.



Problem 8. Find the antiderivative family for the function defined by the formula

$$f(x) = \frac{x+4}{x^2+2x+5}$$

Hint: Complete the square on the denominator, then make an appropriate trigonometric substitution.

Problem 9. The St. Louis Arch takes the form of a *flattened catenary curve* defined by the formula

$$h = f(x) = 34.4(e^{0.01x} + e^{-0.01x}) - 68.8$$

where we take $-300 \leq x \leq 300$. The actual curve opens down, of course.) What is the arc-length of the St. Louis Arch?

Problem 10. For nonzero x and any nonnegative integer n, consider the function

$$I_n(x) = \int_{-1}^{1} (1 - z^2)^n \cos(xz)\, dz$$

Team 1 showed that $xI_0(x) = 2\sin(x)$ and $x^3 I_1(x) = 4\sin(x) - 4x\cos(x)$. Team 2 showed that, for $n \geq 2$, we have

$$x^2 I_n(x) = 2n(2n-1)I_{n-1}(x) - 4n(n-1)I_{n-2}(x)$$

For all nonnegative integers n, let $J_n(x) = x^{2n+1} I_n(x)$. For $n = 0,1,2,3,4$ and 5, verify that

$$J_n(x) = n! \cdot (P_n(x)\sin(x) + Q_n(x)\cos(x))$$

where P_n and Q_n are polynomials with integer coefficients having degree at most n. (This is actually true for *all* nonnegative integers n.)

Problem 11. Find the antiderivative family for the function $y = f(x) = \sin^2(x)\cos^2(x)$.

Problem 12 . It is well-known that Euler's constant e can be represented by the series

$$e = \sum_{j=0}^{\infty} \frac{1}{j!}$$

Suppose that e is rational. That is, suppose $e = p/n$, where p and n are positive integers. Since e is not an integer, we can assume $n \geq 2$; and it follows that $n! \cdot e$ is an integer. In their Problem 10, Team 4 showed that

$$\sum_{j=1}^{\infty} \frac{1}{(n+1)(n+2)\cdot \cdots \cdot (n+j)} < 1$$

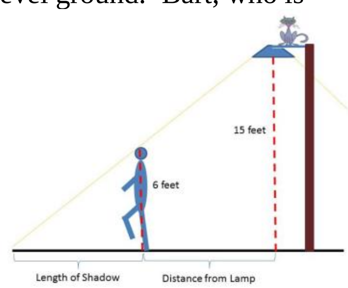
Use this fact and the series representation for $n! \cdot e$ to obtain a contradiction.

Problem 13. What is the Taylor series expansion centered at $x = 2$ for the function $f(x) = e^{3x}$?

Problem 14. Solve the differential equation $t \cdot \frac{du}{dt} + 2u = t^3$, if $u(1) = 0$ and we assume $t > 0$.

Problem 15. If you draw ten straight lines in the plane, no two of them being parallel and no three of them meeting at a point, how many intersection points will there be altogether?

Problem 16. Justo's friend, Bart Vader, is looking for Mr. Hispickles. Bart starts under a street lamp that is mounted 15 feet high on a pole that rises vertically from a stretch of level ground. Bart, who is six feet tall, walks in a straight line away from the base of the pole. When Bart is 40 feet from the pole, the horizontal distance x between Bart and the lamp (measured in feet) is increasing at 5.6 feet per second. How fast is the length of Bart's shadow (measured in feet) changing at this moment?



Problem 17. Consider the matrices shown below.

$$A = \begin{pmatrix} 1 & -3 \\ 0 & 2 \\ -2 & 4 \end{pmatrix} \qquad B = \begin{pmatrix} 3 & -3 & 1 \\ 0 & 3 & -1 \end{pmatrix} \qquad C = \begin{pmatrix} m \\ n \end{pmatrix} \qquad D = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

Determine the solution sets for the systems. Carefully explain your strategies.

$$ABD = \begin{pmatrix} 2 \\ -1 \\ -5 \end{pmatrix} \qquad BAC = \begin{pmatrix} 0 \\ 28 \end{pmatrix}$$

Problem 18. Solve the following system of equations if possible.

$$\begin{cases} (x-y-z)^{-1} + (y-2x)^{-1} - (x-2y)^{-1} = 1 \\ 2(x-y-z)^{-1} + 3(y-2x)^{-1} - 2(x-2y)^{-1} = 3 \\ (x-y-z)^{-1} + (y-2x)^{-1} - 2(x-2y)^{-1} = 0 \end{cases}$$

Problem 19. Construct the slope field for the differential equation $y' - xy = x^2$ on the variable intervals $-3 \leq x \leq 3$ and $-3 \leq y \leq 3$. Use this field to sketch the solution to this differential equation that passes through the point (1,1).

Problem 20. Find the particular solution to the initial value problem shown below.

$$\frac{dy}{dx} = \frac{x}{2-2y} \qquad y(0) = 3$$

Problem 21. Solve the differential equation $y'' - 4y' - 12y = \sin(2x)$.

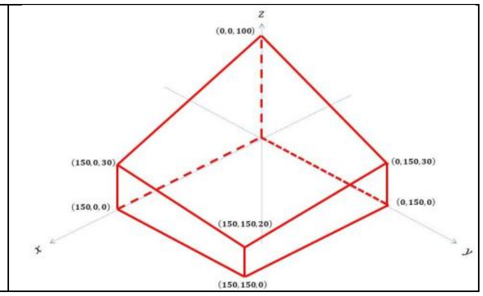
Problem 22. Consider the vector space $\mathcal{R}^3 = (\mathbb{R}^3, +, \cdot)$. Is the following collection of vectors linearly independent in this vector space? Under the Euclidean inner product, what is an orthonormal basis for the subspace spanned by these vectors?

$$S = \left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 8 \\ 3 \\ 8 \end{pmatrix}, \begin{pmatrix} 6 \\ -1 \\ 2 \end{pmatrix} \right\}$$

Problem 23. The sanctuary for a large church is built in the shape of a truncated parallelepiped as shown below. (All measurements are in feet.)

Part (a). Find a formula for the plane that represents the roof of this sanctuary.

Part (b). Use an iterated integral to compute the volume of the sanctuary.



Problem 24. What is the value of the following iterated integral?

$$\int_0^2 \int_0^{\pi/2} \int_0^{1-x} x\cos(y)\,dz\,dy\,dx$$

Problem 25. Consider the linear transformation $T : \mathcal{R}^3 \to \mathcal{R}^3$ defined by the formula

$$T\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 2a - c \\ b + 3a + c \\ 2c - 5b \end{pmatrix}$$

Determine the matrix representation for this transformation with respect to the basis

$$B = \left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} \right\}$$

Problem 26. Consider the linear transformation $T : \mathcal{R}^3 \to \mathcal{R}^3$ defined by the formula

$$T\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} -b+c \\ -a-b \\ -b \end{pmatrix}$$

Part (a). Determine the eigenvalues for this linear transformation.

Part (b). Determine a spanning set for the eigenspace corresponding to each eigenvalue.

Problem 27. Let $\mathbb{Z}$ denote the set of integers, and let $\boldsymbol{\mathcal{Z}} = (\mathbb{Z}, +)$ be the group of integers under the operation of integer addition. Let n be a fixed positive integer and let $n\mathbb{Z}$ denote the set of all integer multiples of n.

Part (a). Prove that $n\mathbb{Z}$ is a cyclic subgroup of the group $\boldsymbol{\mathcal{Z}}$.

Part (b). Prove that $\boldsymbol{\mathcal{Z}}$ is isomorphic to its subgroup $n\mathbb{Z}$.

Problem 28. Let n be a fixed positive integer and let $\boldsymbol{\mathcal{Z}}_n = (\mathbb{Z}_n, \boxplus_n)$ represent the group composed of the set $\mathbb{Z}_n = \{0,1,2,\dots,n-1\}$ under the operation $\boxplus_n$ of addition modulo n.

Part (a). Prove that the function $f : \mathbb{Z} \to \mathbb{Z}_n$ defined by $f(a) = a \bmod(n)$ is a group homomorphism.

Part (b). What is the kernel of this homomorphism?

Let $\mathbb{M}_2$ represent the set of all 2×2 matrices with real number entries. The algebra $\mathfrak{M}_2 = (\mathbb{M}_2, +, *)$ forms a noncommutative ring with unity, where + denotes matrix addition and * denotes matrix multiplication.

Problem 29. Consider the set

$$\mathbb{M}_F = \left\{ \begin{pmatrix} a & b \\ -b & a \end{pmatrix} : a, b \in \mathbb{R} \right\}$$

Part (a). Show that this set forms a subring of the ring $\mathfrak{M}_2$. Is this set an ideal of $\mathfrak{M}_2$?

Part (b). Show that $(\mathbb{M}_F, +, *)$ is actually a field.

Problem 30. Let $X = \{a, b, c, d, e\}$ and let $Y = \{u, v, w, x\}$ equipped with the following topologies.

$$\Omega_X = \{\emptyset, X, \{a\}, \{b\}, \{a,b\}, \{d,e\}, \{b,d,e\}, \{a,d,e\}, \{a,b,d,e\}\}$$
$$\Omega_Y = \{\emptyset, Y, \{u\}, \{x\}, \{u,x\}, \{v,x\}, \{u,v,x\}\}$$

Is the function $f : X \to Y$ defined by the tabular notation below continuous relative to these topologies?

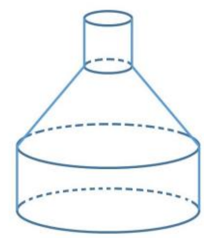
$$f : \begin{pmatrix} a & b & c & d & e \\ u & u & w & v & v \end{pmatrix}$$

Problem 31. If $z = \cos(\theta) + i\sin(\theta)$, then what is the complex representation of the power z^n for any positive integer n under complex multiplication? Justify your answer with a formal proof.

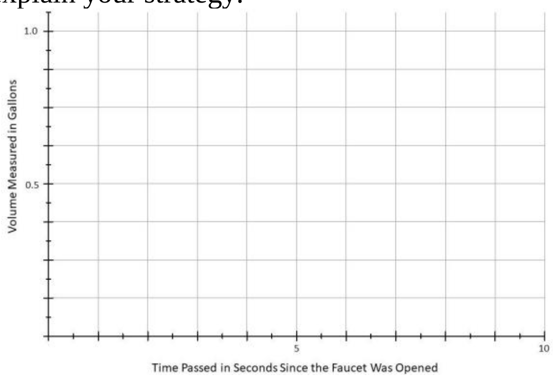
TEAM 4

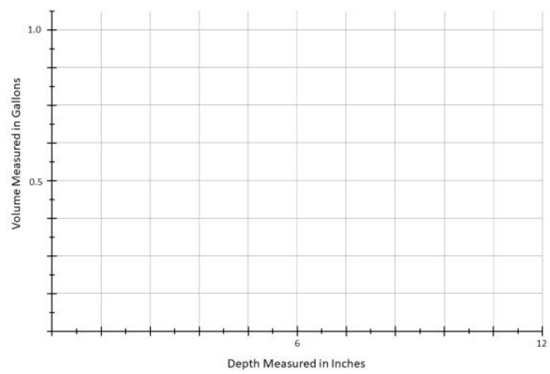
Problem 1. Wilhelmina purchased a glass beaker at an antique shop. The beaker is twelve inches tall and is capable of holding one gallon of liquid. Wilhelmina places the beaker under a faucet and begins to fill it. Assume water is entering the beaker at a constant rate of 0.10 gallons per minute.

- Let Let t represent measures of the quantity "time passed since the faucet was turned on, measured in seconds."
- Let V represent measures of the quantity "the volume of water in the beaker, measured in cubic inches."
- Let h represent measures of the quantity "the depth of water in the beaker, measured in inches."

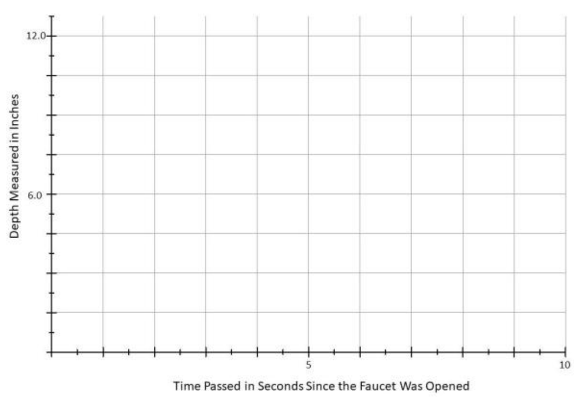


Part (a). Using the grids below, sketch traces that you think would reasonably describe the function $V = f(t)$ in the interval $0 \leq t \leq 10$ and the function $V = g(h)$ in the interval $0 \leq h \leq 12$. Be prepared to explain your strategy.





Part (b). Use the grid below to sketch a trace that could reasonably describe the function $h = m(t)$ in the interval $0 \leq t \leq 10$. Be prepared to explain your strategy.

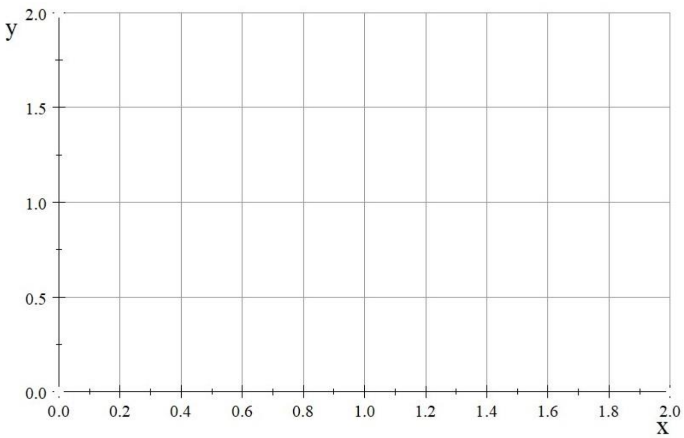


Problem 2. In 1975, the population of Mexico was 84 million and was growing at a continuous annual rate of 2.6%, while the population of the United States was 250 million and growing at a continuous annual rate of 0.7%. If we compare instantaneous growth rates in terms of people per year, which population was growing faster in 1975?

Problem 3. What is the third derivative of the function $y = f(x) = \text{Arcsin}(x)$?

Problem 4. On the grid provided, sketch the graph of a function $y = f(x)$ that satisfies the following properties.

- $f''(x) > 0$ on the input intervals $0 \le x < 0.8$ and $1.4 < x \le 2$
- $f''(x) < 0$ on the input interval $0.8 < x < 1.4$
- $f'(x) < 0$ on the input intervals $0 \le x < 0.6$ and $1 < x < 1.6$
- $f'(x) > 0$ on the input intervals $0.6 < x < 1$ and $1.6 < x \le 2$



Problem 5. Are there any ordered pairs (a, b) where the tangent line to the graph of $xy = x^2 - y^3$ is undefined? If so, what are these ordered pairs?

Problem 6. Consider the function defined by the following formula.

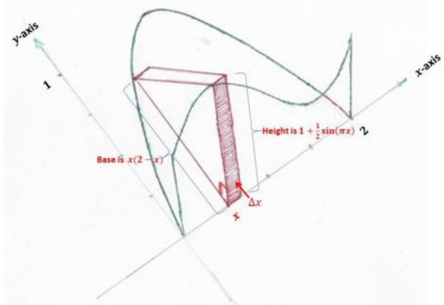
$$f(x) = \begin{cases} 0 & \text{if} \quad x = 0 \\ x^3 \cos\left(\dfrac{1}{x}\right) & \text{if} \quad x \neq 0 \end{cases}$$

Is this function differentiable at $x = 0$? If so, what is the value of $f'(0)$?

Problem 7. Evaluate the definite integral shown below.

$$\int_0^{\pi/4} e^x \sin(2x)\, dx$$

Problem 8. What is the volume of the solid obtained by revolving the region R about the y-axis, where R is bounded by the curves $x = 1$, $x = \sqrt{3}$, $y = f(x) = \text{Arctan}(x)$, and $y = g(x) = \ln(x)$?



Problem 9. A solid has a base in the xy-plane that is the region enclosed between $x = 0$, $x = 2$, and $y = f(x) = x(2 - x)$. Cross sections of this solid taken perpendicular to the xy-plane and parallel to the y-axis are right triangles whose vertical leg lies on the x-axis. The heights of the triangles are determined by the formula

$$y = f(x) = 1 + \frac{1}{2}\sin(\pi x)$$

(See the figure on the right.) Determine the volume of this solid.

Problem 10. For nonzero x and any nonnegative integer n, consider the function

$$I_n(x) = \int_{-1}^{1} (1 - z^2)^n \cos(xz)\, dz$$

Part (a). Prove that $0 < I_n(\pi/2) \leq 2$.

Part (b). Now, suppose that $\pi/2$ is *rational*; that is, suppose $\pi/2 = a/b$ where a and b are positive integers. Use the formula

$$x^{2n+1} I_n(x) = n! \cdot (P_n(x)\sin(x) + Q_n(x)\cos(x))$$

(where P_n and Q_n are polynomials with integer coefficients having degree at most n) to show that

$$\frac{a^{2n+1}}{n!} I_n\left(\frac{\pi}{2}\right) = P_b\left(\frac{\pi}{2}\right) \cdot b^{2n+1}$$

Part (c). Explain why the expression in Part (b) is a positive integer.

Part (d). Explain why we know

$$\lim_{n\to\infty} \frac{a^{2n+1}}{n!} I_n\left(\frac{\pi}{2}\right) = 0$$

Use this fact to explain why we must conclude that π is *irrational*. (This proof was first presented by Mary Cartwright in 1945.)

Problem 11. Let n be a positive integer larger than 1. Show that the following statements are true.

$$\sum_{j=1}^{\infty} 2^{-j} = 1 \qquad\qquad \sum_{j=1}^{\infty} \frac{1}{(n+1)(n+2)\cdot \cdots \cdot (n+j)} < 1$$

Hint: Note that $n + k > 2$ for $1 \leq k \leq j$.

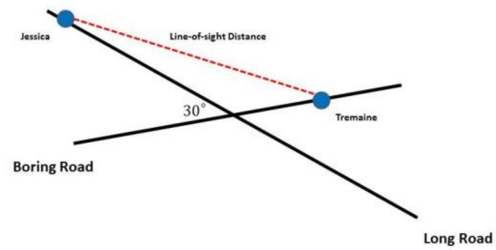
Problem 12. A printing company needs to make a rectangular poster for the new thriller "It Came from the Coffee Pot." The printed matter must have a six-inch margin on the bottom, a four-inch margin on the top, and three-inch margins on each side. If the printed matter requires an area of 1,000 square inches, what dimensions will minimize the area of the poster?

Problem 13. What is the fourth-degree Maclaurin polynomial for the function $f(x) = \sin(x)\cos(x)$?

Problem 14. Solve the differential equation $4p^3 \cdot \frac{dp}{dv} = e^{2v}$.

Problem 15. A folder contains ten copies each of versions *A*, *B*, *C*, *D*, and *E* of a college algebra final exam. If Lola pulls exams out of the folder at random, how many will she have to pull out in order to be guaranteed that she has five of the same version ?

Problem 16. Long Road and Boring Roads are both straight for several miles and intersect at a 30° angle as shown below. Tremaine is driving away from the intersection on Boring Road at a constant speed of 20 miles per hour while Jessica is driving toward the intersection on Long road at a constant speed of 45 miles per hour. How fast is the line-of-sight distance between Jessica and Tremaine changing when Jessica is 1.5 miles from the intersection and Tremaine is 0.75 miles from the intersection?



Problem 17. If a square matrix A is singular, is it also true that its transpose A^T is singular? What about the matrix products AA^T and $(A^TA)^T$?

Problem 18. Is the system of linear equations shown below dependent or independent? Determine its solution set.

$$\begin{cases} 2x & - & y & + & 3z & + & w & = & 2 \\ x & + & 3y & - & z & + & w & = & 0 \\ & & -7y & + & 5z & - & w & = & 2 \\ 4x & - & 9y & + & 11z & + & w & = & 6 \end{cases}$$

Problem 19. Consider the differential equation shown below.

$$\frac{dy}{dx} = \frac{xy}{1-y^2}$$

Part (a). Construct a slope field for the differential equation shown below, then use your slope field to approximate the graph of the solution that passes through the point $(2,2)$.

Part (b). Is it possible to determine the actual formula for this solution? Justify your answer.

Problem 20. Solve the initial value problem $y'' - 8y' + 17y = 0$, $y(0) = -4$ and $y'(0) = -1$.

Problem 21. Suppose that f is the step function defined on the interval $[0,1]$ by the rule

$$f(x) = \begin{cases} 0 & \text{if} & x = 0 \\ 2^{1-n} & \text{if} & 2^{-n} < x \le 2^{1-n}\ (n \in \mathbb{Z}^+) \end{cases}$$

Does $\int_0^1 f(x)dx$ exist? If so, what is its value?

Problem 22. Consider the vector space $\mathcal{R}^4 = (\mathbb{R}^4, +, \cdot)$ with the usual vector addition $+$ and scalar multiplication $\cdot$ over the field of real numbers. Give an elementwise description of the vectors contained in the subspace spanned by the set

$$S = \left\{ \begin{pmatrix} -1 \\ 3 \\ 0 \\ 4 \end{pmatrix}, \begin{pmatrix} 2 \\ -3 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \\ 2 \\ 12 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \\ 3 \\ 0 \end{pmatrix} \right\}$$

Problem 23. What are the saddle points for the function $f(x, y) = x^2 - 4y^2$?

Problem 24. A heating element is placed beneath a circular metal plate of radius five centimeters, and the plate starts to warm. Let x and y represent the east-west and north-south distances, respectively from the center of the plate, measured in centimeters, and suppose the temperature in degrees Celsius of any point on the plate is given by the formula $C = T(x, y) = 150 - (4x^2 - (y - 2)^2)$. What is the direction of maximum temperature increase from the point $(2, -3)$?

Problem 25. Consider the linear transformation $T : \mathcal{R}^2 \to \mathcal{R}^3$ having the property that

$$T\begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 5 \\ 3 \end{pmatrix} \quad T\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -2 \\ 4 \end{pmatrix}$$

What is the value of $T\begin{pmatrix} 3 \\ 1 \end{pmatrix}$?

Problem 26. Use eigenvalues to solve the following system of differential equations.

$$\begin{cases} \dfrac{du}{dt} = 4u + w \\[2mm] \dfrac{dv}{dt} = 2u + 3v + 2w \\[2mm] \dfrac{dw}{dt} = u + 4w \end{cases}$$

Let $U_{2,2}$ represent the set of all nonsingular (invertible) 2×2 matrices with real number entries. If we let $*$ represent 2×2 matrix multiplication, then $\boldsymbol{GL_2} = (U_{2\times 2}, *)$ forms a group (called the *General Linear Group*). The identity for this group is the 2×2 identity matrix, and for each $A \in U_{2\times 2}$, we have

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \Rightarrow \quad A^{-1} = \frac{1}{\mathrm{Det}(A)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Problem 27. Consider the set

$$Z = \left\{ \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} : a \in \mathbb{R}' \right\}$$

Part (a). Prove that Z is a subgroup of the General Linear Group.

Part (b). If $A \in Z$ and $M \in U_{2\times 2}$, prove that $A * M = M * A$.

Part (c). Explain why Part (b) allows you to conclude that Z is a normal subgroup of $\boldsymbol{GL_2}$.

Consider the group $\mathcal{S}_4$ of permutations on the four-element set $X = \{1,2,3,4\}$ under the operation of function composition. Let A_4 be the subcollection

$$E = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix} \quad G = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 1 & 4 \end{pmatrix} \quad H = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 2 & 4 \end{pmatrix} \quad I = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix}$$

$$J = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 2 & 3 \end{pmatrix} \quad K = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 2 & 1 & 3 \end{pmatrix} \quad L = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix} \quad M = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 3 & 2 \end{pmatrix}$$

$$N = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 4 & 2 \end{pmatrix} \quad O = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix} \quad P = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 4 & 1 \end{pmatrix} \quad Q = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 3 & 1 \end{pmatrix}$$

The set A_4 forms a subgroup of the group $\mathcal{S}_4$; its operation table is provided below.

∘	*E*	*G*	*H*	*I*	*J*	*K*	*L*	*M*	*N*	*O*	*P*	*Q*
E	*E*	*G*	*H*	*I*	*J*	*K*	*L*	*M*	*N*	*O*	*P*	*Q*
G	*G*	*H*	*E*	*P*	*Q*	*O*	*J*	*K*	*I*	*M*	*N*	*L*
H	*H*	*E*	*G*	*N*	*L*	*M*	*Q*	*O*	*P*	*K*	*I*	*J*
I	*I*	*J*	*K*	*E*	*G*	*H*	*O*	*P*	*Q*	*L*	*M*	*N*
J	*J*	*K*	*I*	*M*	*N*	*L*	*G*	*H*	*E*	*P*	*Q*	*O*
K	*K*	*I*	*J*	*Q*	*L*	*O*	*N*	*L*	*M*	*H*	*E*	*G*
L	*L*	*M*	*N*	*O*	*P*	*Q*	*E*	*G*	*H*	*I*	*J*	*K*
M	*M*	*N*	*L*	*J*	*K*	*I*	*P*	*Q*	*O*	*G*	*H*	*E*
N	*N*	*L*	*M*	*H*	*E*	*G*	*K*	*I*	*J*	*Q*	*O*	*P*
O	*O*	*P*	*Q*	*L*	*M*	*N*	*I*	*J*	*K*	*E*	*G*	*H*
P	*P*	*Q*	*O*	*G*	*H*	*E*	*M*	*N*	*L*	*J*	*K*	*I*
Q	*Q*	*O*	*P*	*K*	*I*	*J*	*H*	*E*	*G*	*N*	*L*	*M*

Problem 28. Consider the subgroup $\Theta = \{E, I, L, O\}$ of the group $(A_4,\circ)$.

Part (a). Construct the family RC_Θ of right cosets for the subgroup Θ.

Part (b). Show that Θ is a normal subgroup of the group $(A_4,\circ)$.

Part (c). Construct the operation table for the quotient group A_4/Θ.

Part (d). Show that this quotient group is isomorphic to the group $\boldsymbol{\mathcal{Z}_3}$.

Let $\mathbb{M}_2$ represent the set of all 2×2 matrices with real number entries. The algebra $\mathfrak{M}_2 = (\mathbb{M}_2, +, *)$ forms a noncommutative ring with unity, where + denotes matrix addition and * denotes matrix multiplication.

Problem 29. Consider the ring $\mathfrak{M}_2$.

Part (a). Show by example that this ring contains zero-divisors.

Part (b). Show by example that this ring contains infinitely many units.

Problem 30. Suppose that a set X is endowed with two topologies, call them Ω_1 and Ω_2. Which of the following collections of subsets will also be a topology on X?

$$\Omega = \{A \cup B : A \in \Omega_1 \text{ and } B \in \Omega_2\} \qquad \tau = \Omega_1 \cap \Omega_2 \qquad \theta = \{A \cap B : A \in \Omega_1 \text{ and } B \in \Omega_2\}$$

Problem 31. In the arena of complex arithmetic, it is customary to let $e^{i\theta}$ represent the complex number $z = \cos(\theta) + i\sin(\theta)$. Note that the set of all such numbers constitute the unit circle in the plane. Using this notation, how could an *arbitrary* complex number be represented?