A *binary operation* on a set X is a two-variable function that takes any two members of the set X as input and returns another member of the set X as output. The rule may be defined by a table, a formula, or even a graph.

Mathematicians usually consider the following two statements to mean the same thing:

- A particular binary combining rule is an operation on a set X.
- A set X is closed under a particular binary combining rule *and* the rule defines a two-variable function.

A set X endowed with one or more binary operations is an example of an *algebra*. We usually represent an algebra as an n-tuple in which the first coordinate is the set (known as the *universe* of the algebra), and the remaining coordinates list symbols for the operations.

For example, the set $\mathbb{Z}$ of integers[1] is closed under (integer) addition $+$ and multiplication $\cdot$; hence we consider $\mathbb{Z}$ to be an algebra under these two rules. In these notes, we will let $\mathfrak{Z} = (\mathbb{Z}, +, \cdot)$ represent this algebra.

Problem 1. Does the set $\mathbb{Q}$ of rational numbers form an algebra under any of the following rules? Explain your thinking on each rule. (In each case, assume $p, q, a,$ and b are integers, and assume that $q, b \neq 0$.)

$$\frac{p}{q} \ltimes \frac{a}{b} = \frac{qb}{ap} \qquad \frac{p}{q} \asymp \frac{a}{b} = p + a \qquad \frac{p}{q} \divideontimes \frac{a}{b} = \sqrt[3]{\left(\frac{p}{q}\right)^3 \left(\frac{a}{b}\right)^6}$$

It is common to use tables to define binary rules on small sets, and we have certain conventions that we use when interpreting such tables. For example, suppose we let $X = \{1,2,3,4\}$ and let $\propto$ be the binary rule defined by the following table.

$\propto$	**1**	**2**	**3**	**4**
1	2	1	4	2
2	3	4	4	1
3	4	1	3	2
4	4	4	2	2

This table is read from left-to-right. In other words, when we consider the term $2 \propto 3$, it is understood that we are looking at the intersection of the "2-row" and the "3-column". Therefore, based on this table,

$$2 \propto 3 = 4$$

Problem 2. Is the set $X = \{1,2,3,4\}$ an algebra under the rule $\propto$? Explain your thinking.

[1] The symbol $\mathbb{Z}$ is the first letter of the German word "*zahle*" which means "number."

Commutative Binary Operation on a Set

Let X be a nonempty set, and suppose that $\star$ represents a binary operation on the set X. We say that two elements a and b from the set X *commute* provided $a \star b = b \star a$.

We say that the binary operation $\star$ is *commutative* provided $a \star b = b \star a$ for *every* pair of elements $a, b \in X$.

Problem 3. Let $\mathbb{R}^+$ represent the set of all positive real numbers, and consider the binary operations $\star$ and $\curlywedge$ defined on $\mathbb{R}^+$ by the formulas

$$a \star b = e^{a/b} \qquad \text{and} \qquad a \curlywedge b = e^{ab}$$

Part (b). Is either operation commutative? Justify your answer.

Part (c). Does it make sense to say that the set $\mathbb{R}^+$ is commutative? Explain.

Problem 4. Let $X = \{a, b, c, d\}$ and consider the binary operations $\star$ and $\curlywedge$ defined by the tables below. Are either of these operations commutative? Explain your thinking.

$\star$	$\boldsymbol{a}$	$\boldsymbol{b}$	$\boldsymbol{c}$	$\boldsymbol{d}$
$\boldsymbol{a}$	c	d	a	b
$\boldsymbol{b}$	d	c	b	a
$\boldsymbol{c}$	a	b	c	d
$\boldsymbol{d}$	b	a	d	c

$\curlywedge$	$\boldsymbol{a}$	$\boldsymbol{b}$	$\boldsymbol{c}$	$\boldsymbol{d}$
$\boldsymbol{a}$	b	a	d	b
$\boldsymbol{b}$	a	d	d	a
$\boldsymbol{c}$	d	a	c	b
$\boldsymbol{d}$	d	d	b	b

Associative Binary Operation on a Set

Let X be a nonempty set, and suppose that $\star$ represents a binary operation on the set X. We say that the binary operation $\star$ is *associative* provided

$$a \star (b \star c) = (a \star b) \star c$$

for *all* elements $a, b, c \in X$.

Determining whether or not a given binary operation is associative can be challenging; consequently, we often focus on operations built from ones we know are associative.

Familiar Properties of Real Number Arithmetic

We will assume that real number addition and multiplication satisfy all of the arithmetic properties you are familiar with. In particular, we will assume that real number addition and multiplication are commutative and associative. We will also assume that real number multiplication distributes over real number addition.

Problem 5. Let $\mathbb{Z}$ represent the set of integers. Construct a proof to show that the binary operation on the set $\mathbb{Z}$ defined by

$$a \diamond b = a + b - ab$$

is associative. Is this operation commutative?

Homework.

Problem 1. None of the binary combining rules below is a binary operation on the set of rational numbers. What is wrong with each rule? (In each case, assume $p, q, a,$ and b are integers, and assume $q, b \neq 0$.)

(a) $\frac{p}{q} \sqcap \frac{a}{b} = \arcsin\left(\frac{pa}{qb}\right)$ (b) $\frac{p}{q} \propto \frac{a}{b} = \sqrt[3]{\frac{p}{q} + \frac{a}{b}}$ (c) $\frac{p}{q} \wedge \frac{a}{b} = qb$

Problem 2. Consider the combining rule defined by $x \between y = (x^2 + y^2)^{1/2}$. Is this rule a binary operation on the set $\mathbb{Z}^+$ of positive integers? Justify your answer.

Problem 3. Is either of the combining rules below a binary operation on the set of nonzero integers? Explain.

(a) $a \eqcirc b = d$ where d is the largest integer factor of ab that is not equal to ab.

(b) $a \triangleq b = xy$ where x and y are both integer factors of ab.

Problem 4. Consider the binary operation $\|$ on the set $\mathbb{Z}$ of integers defined by $a \parallel b = |a + b|$.

Part (a). Prove that this operation is commutative.

Part (b). Give a specific counterexample to show that this operation is *not* associative.

Problem 5. Consider the binary operation $\Bumpeq$ on the set $\mathbb{R}$ of real numbers defined by

$$x \Bumpeq y = (x^3 + y^3)^{1/3}$$

Prove that this operation is associative.

Let $\mathbb{R}$ represent the set of real numbers, and let $\mathrm{Lin}[\mathbb{R} \to \mathbb{R}]$ denote the set of all *linear* functions on $\mathbb{R}$. That is, $f \in \mathrm{Lin}[\mathbb{R} \to \mathbb{R}]$ if and only if $f : \mathbb{R} \to \mathbb{R}$ and $f(x) = mx + b$ for some real numbers m and b.

Problem 6. Consider the binary combining rule $f \circ g$ defined by $[f \circ g](x) = f\big(g(x)\big)$. (In other words, consider the combining rule defined by function composition.)

Part (a). Show that $f \circ g$ is a member of $\mathrm{Lin}[\mathbb{R} \to \mathbb{R}]$ for all $f, g \in \mathrm{Lin}[\mathbb{R} \to \mathbb{R}]$. (In other words, show that $f \circ g$ is also a linear function.) This tells us that $\mathrm{Lin}[\mathbb{R} \to \mathbb{R}]$ is closed under function composition.

Part (b). Part (a) tells us that function composition is a binary operation on $\mathrm{Lin}[\mathbb{R} \to \mathbb{R}]$. Is this operation commutative? Justify your answer.

Part (c). Is this operation associative? Justify your answer.